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A structural derivation of the option price

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Can bailout improve the economic welfare?

A structural derivation of the option price

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Abstract

I succeed in deriving the Black-Scholes formula from the payoff functions within a some kind of zero-sum game. Such a structural approach enables us to apply the formula to a more general case than the one presented in the original paper. The most relevant case concerns the bailout policy, which is weaved into investors' rational expectations. Once such a policy is anticipated, the price of option becomes dear and investors' behavior becomes more bullish since the bailout policy eliminates loss when speculation fails.

Since the transaction of derivatives is generically a zero-sum game, the bailout policy never improves economic welfare. Indeed, when we consider that the financial source of the bailout policy is levied from other economic agents, it becomes apparent that such a policy would surely lower economic welfare.

1 Introduction

It is not self-evident whether bailout policies that are often adopted in financial crises improves economic welfare. This is mainly because payoff functions of financial assets are not explicitly defined. This article provides the payoffs function of a European call option explicitly and considers the welfare implication of the bailout policy. As its corollary, I induce the Black-Scholes [1] formula from the payoff functions.

Since the transaction of derivatives is generically a zero-sum game, an anticipated bailout policy does not rescue its traders. Once such a bailout policy is rationally weaved into expectations, it becomes a part of the value of the option. Hence, the bailout policy does not affect investors' risk-neutral expected utility. However, the outsiders of the derivative transactions are heavily levied for financing such a policy, and thus, economic welfare, as a whole, is always worsened by the policy.

The paper consists of three sections. In Section 2, I define the payoff functions of a European call option and analyzes their properties. Section 3 derives the Black-Scholes formula to ascertain the validity of my approach. Using the obtained results, Section 4 solves the option price that is attached by the money poured into the bailout policy, and considers the macroeconomic welfare implication of the policy. In Section 5, we provide brief concluding remarks.

2 The Model

I consider a European call option the expiry date is T. A European call option is a kind of zero-sum game that follows the concept of the Stackerberg equilibrium. The strategy of a buyer (follower) relates to price at which she/he exercises the option. The seller's strategy (leader) is to offer the price of option and its exercise price.

As such, we obtain the following theorem.

Theorem 1 *Let us denote the buyer's and seller's payoff functions as* V_t^B *and* V_t^S*, respectively. They can be written as*

$$V_t^B(t:T) = E_t[\max[e^{-r[T-t]}[S_T - X] - P_t, -P_t]], \tag{1}$$

$$V_t^S(t:T) = E_t[\min[e^{-r[T-t]}[X - S_T] + P_t, P_t]], \tag{2}$$

where P_t is the option price at time t, S_T is the stock price at the expiry date T, and X is the exercise price. E_t denotes the conditional expectation operator on the available information until time t.

To prove the above theorem, the following lemma is of use.

Lemma 1 *Early exercise does not occur with probability one.*

Proof.

Suppose that an early exercise occurs at time $\tilde{t}$ ($t < \tilde{t} < T$) with some positive probability. Then,

$$V^B(t:T) \geq e^{-r[\tilde{t}-t]} \int_{[S_{\tilde{t}} - X]e^{-r[\tilde{t}-t]} \geq P_t} [S_{\tilde{t}} - X] d\Phi(S_{\tilde{t}} \mid S_t) - P_t > 0 \tag{3}$$

holds. $\Phi(\cdot)$ is the conditional cumulative distribution function of $P_{\tilde{t}}$ on P_t. The above inequality exists given the fact that an early exercise occurs because of the chance of the excess gain.

On the other hand, from (3), the seller's payoff function satisfies

$$V^S(t:T) \leq P_t - e^{-r[\tilde{t}-t]} \int_{[S_{\tilde{t}} - X]e^{-r[\tilde{t}-t]} \geq P_t} [S_{\tilde{t}} - X] d\Phi(S_{\tilde{t}} \mid S_t) < 0. \tag{4}$$

From (4), the seller's payoff becomes negative whenever an early exercise occurs with some positive probability, and hence such an asset would never be provided. Accordingly, any early early exercise never occurs with positive probability.

Proof of Theorem 1.

From Lemma 1, since the call option is held till the expiry date T with probability one, the values of the call option become the expected discount values at T.

From Theorem 1, the following theorem holds.

Theorem 2 *The equilibrium price of the call option* P_t *is expressed as*

$$P_t = E_t[\max[e^{-r[T-t]}[S_T - X], 0]]. \tag{5}$$

To prove Theorem 2, the following lemma is of use.

Lemma 2

$$V^S(t:T) = V^B(t:T) = 0 \tag{6}$$

holds.

Proof.

By adding up (1) and (2), we obtain

$$V^S(t:T) + V^B(t:T) = 0.$$

Since both V^S and V^B are non-negative, (6) holds.

The proof of Theorem 2

Substituting (6) into (1) and (2), I can obtain (5).

3 The Black-Scholes Formula

In this section, I derive the Black-Scholes formula based on the structural approach explained above.

When the stock price S_t follows the geometric Brownian motion with drift r and instantaneous variance σ^2, by Ito's formula, the logarithm of the stock price s_t follows the normal distribution, Ψ, the mean and standard deviation of which are $[[r-\frac{\sigma^2}{2}][T-t], \sigma\sqrt{T-t}]$. In addition, it can be easily shown by some elementary calculus that

$$\int_X^\infty S_T d\Psi(S_T \mid S_t) = S_t e^{-r[T-t]}\Phi^{SN}(z_t \geq -z^*), \tag{7}$$

$$z_T \equiv \frac{s_T - s_t - [r+\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}, \quad z^* \equiv \frac{\ln\frac{S_t}{X} + [r+\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}},$$

where Ψ^{SN} is the standard normal cumulative distribution function.

Since the logarithmic value s_t of the stock price S_t follows $N([r-\frac{\sigma^2}{2}][T-t], \sigma\sqrt{T-t})$, $y_T \equiv \frac{s_T - s_t - [r-\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}$ also follows the standard normal distribution Φ^{SN}. Let us define the critical value y^* as

$$y^* \equiv \frac{\ln\frac{S_t}{X} + [r-\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}.$$

Substituting these results into (5), we obtain

$$P_t = S_t \cdot \Pr(z_T \geq -z^*) - X \cdot \Pr(y_T \geq -y^*).$$

Finally, by the symmetry of the normal standard distribution,

$$P_t = S_t\Phi^{SN}(z^*) - X\Phi^{SN}(y^*). \tag{8}$$

This is the Black-Scholes formula.

4 The Welfare Economic Implication of the Bailout Policy

Usually, the validity of the bailout policy is judged from the view of income distribution, that is, whether the redistribution of incomes from tax payers to failed speculators is legitimate. Instead, in this paper, I deal with the efficiency of the bailout policy. This is not a self-evident problem because some parts of money of an anticipated bailout will be consumed to stimulate the zero-sum game within the option trading. This is because that the bailout guarantees the minimum return for buyers and causes a kind of moral hazard which is intrinsic to the property of limited liability. Thus, not all the money poured into the bailout cannot be used for the compensation of the capital loss, and thus, the effect of the bailout policy is rather mitigated. Consequently, as I have detailed below , the bailout policy acts conversely and worsens economic welfare. Besides advancing the disparity of income distribution, the bailout policy harms peoples' well-being.

To endorse the above discussion, let us define the payoff functions under a bailout policy. The assumed bailout policy is as follows: at expiry date T , money, which is levied by outsiders and amounts to M , is transferred to the losers (i.e., sellers, when the option is exercised and buyers, when not exercised).

We must note that the *effective* exercise price rises from X to $X+M$. This is because the bailout money M becomes the lower bound of the seller's revenue, and thus, the option is exercised only when $S_T - X \geq M$.

$$V_t^B(t:T,M) = E_t[\max[e^{-r[T-t]}[S_T - [X+M]] - P_t, e^{-r[T-t]}M - P_t]]$$
$$= e^{-r[T-t]}[\int_{X+M}^{\infty}[S_T - [X+M]]d\Phi(S_T \mid S_t) + M\Phi(X+M)] - P_t. \tag{9}$$

Corresponding to (9), the seller's payoff function becomes

$$V_t^S(t:T,M) = P_t - e^{-r[T-t]}\int_{X+M}^{\infty}[S_T - [X+M]]d\Phi(S_T \mid S_t). \tag{10}$$

Note that when $S_T - X < M$, the option is not exercised. Since there is no loss on the sellerr's side, the term M corresponding to such a case in the above equation does not appear.

Summing up both sides of (9) and (10), I obtain

$$V^B(t:T,M) + V^S(t:T,M) = e^{-r[T-t]}M\Phi(X+M) < e^{-r[T-t]}M. \tag{11}$$

Since the actual total sum of money poured into the bailout is $e^{-r[T-t]}M$, (11) implies that the social cost of the bailout policy exceeds the benefits. Consequently, we reach the following important theorem.

Theorem 3 *The bailout policy always harms the economic welfare.*

The background of the above theorem is as follows. The option is exercised with probability $1-\Phi(X+M)$. In such a case the transaction of the option attached the bailout policy becomes a zero-sum game because the exercise price increases by M, and there is no substantial effect of the bailout policy. In other words, a kind of moral hazard owing to the limited liability occurs on the buyer's side. Since the value function of seller is passively defined in accordance with the buyer's action, an increase in the *effective* exercise price enlarges the seller's loss and there is no social gain in such a case.

Finally, I solve the equilibrium option price using the Black-Sholes formula. There is an indeterminacy concerning the pricing, because the game is not zero-sum and the values of the payoff functions differ in the case of the bailout policy. Thence, I consider the case that the competition among sellers is under strain, and there is no surplus for them (i.e., $V^S(t:T,M)=0$). Then, applying the Black-Scholes formula to (10), I obtain

$$P_t = S_t\Phi^{SN}(z_M^*) - [X+M]\Phi^{SN}(y_M^*), \tag{12}$$

$$z_M^* \equiv \frac{\ln\frac{S_t}{X+M} + [r+\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}, \quad y_M^* \equiv \frac{\ln\frac{S_t}{X+M} + [r-\frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}.$$

Note that $V^B(t:T,M) = e^{-r[T-t]} > 0$, and hence, the buyers are ready to accept such an offer price. Thus, the trading is surely settled.

5 Concluding Remarks

This paper presented a game-theoretic approach concerning option pricing, the validity and tractability of such an approach were ascertained by deriving the Black-Scholes formula. I also applied this approach to welfare implications of the bailout policy. I found that such a policy always worsens economic welfare. This is because of the moral hazardous behavior of the buyer owing to the limited liability which has been previously emphasized in studies such as by Arrow [2] and Stiglitz and Weiss [3].

References

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