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and Negative Intertemporal Externality  
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# Properties of the Social Discount Rate and Negative Intertemporal Externality in the Utility or Production Function

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## Abstract

This study examines the role of the social discount rate, as applied to the efficient allocation of carbon dioxide (CO<sub>2</sub>) emissions, which provokes serious intertemporal negative externalities such as global warming. Two approaches exist in introducing the negative intertemporal externality brought about by such emissions into economic theory. One is the utility-function approach, and the other is the production-function approach. The former is well-suited to the case in which climate change directly threatens the living standard of human, for example, through a drastic degradation of the environment. The latter approach becomes relevant to case in which global warming is expected to lead to productivity stagnation particular to the additional costs of adaptation.

This study shows that both approaches achieve the most efficient allocation in the vicinity of stationary state when a social planner applies a social discount rate close to zero, because the marginal substitution rate of carbon emissions to consumption is correctly reflected in the relative price. In other words, a positive social discount rate underestimates the negative capitalized value of emissions and hence prompts excess consumption and emissions, regardless of which of the two approaches is chosen.

**Keywords:** Utility function with a negative externality; Production function with a negative externality; Social discount rate; Efficiency in emissions allocation

**JEL Classifications:** D61; ;D62; Q30; Q51

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## 1. Introduction

Global warming can be regarded as a type of intertemporal congestion problem, which originates from the abuse of our non-renewable atmosphere. Excess emissions of CO<sub>2</sub>, provoked by unlimited combustion of fossil fuels are a crucial factor in global warming. Theoretically speaking, this can be seen as an application of the club goods theory as follows. The deterioration in the quality of the atmosphere due to excess emissions decreases the service that the atmosphere can provide us. For each individual emitter the CO<sub>2</sub> contribution is negligible, and congestion is as entirely external for them. However, as the foregoing discussion suggests, this congestion phenomenon is internal for the overall global economy. It is particularly serious and complicated is to resolve owing to its dynamic nature. With continued emissions, the concentration of CO<sub>2</sub> in the atmosphere gradually increases over time, leading to more severe global warming. Accordingly, determining methods of evaluating future damage becomes vital, making the issue of the social discount rate highly significant.

In addition, in a market economy, the social discount rate is closely related to carbon tax. This is because the optimal carbon price (tax) should be equalized with the capitalized value of carbon by the social discount rate. One will find how dear the carbon price (tax rate) is and sensitive to the social discount rate.

There is an important distinction over which functions is more such seriously affected by this negative externality: the utility function or the production function. One can argue that such distinction is meaningless and that the negative externality affects both functions. Nevertheless, there are two schools of thought in theoretical environmental economics. One is the production function approach. The typical examples are Nordhaus (2014) and Kuninori and Otaki (2016). The other is the utility function approach (e.g., Uzawa (2009), Otaki (2013, 2015)). It seems an acute issue to analyze whether the implications concerning the optimal social discount rate are dependent on the approach chosen.

This study is an attempt to clarify the following two theoretical issues. One is to explore the property of optimal social discount rate if it exists. The other is to consider whether such properties are affected by the choice of the foregoing two models.

The rest of the study is organized as follows: Chapter 2 constructs an optimal growth model in which the intertemporal negative externality originating from excess CO<sub>2</sub> emissions affects the utility function. The properties of optimal social discount rate and carbon tax are also considered. Chapter 3 transforms the model with the negative externality in the production function, and compares with the results obtained in the former section. Section 4 provides some brief concluding remarks.

## 2. Utility-Function Approach Model

Utility function approaches assumes that the negative intertemporal externality originating from excessive CO<sub>2</sub> emissions directly affects people's utility. It is assumed that a strictly concave instantaneous utility function,  $u_t$ , is represented by

$$u(t) \equiv u(c_t, -E_t) e^{-\rho t}, u_1, u_2 > 0, \quad (1)$$

where  $c_t, E_t$  denote consumption and accumulated emissions, respectively.  $\rho$  is the social discount rate applied to emissions.  $u_i$  is the partial derivative with respect to the  $i$ -th argument of the function.

It is assumed that emissions are partially absorbed mainly by oceans and forests, and also increases proportionately with consumption. Accordingly, the carbon cycle can be described by the following differential equation. That is,

$$\dot{E}_t = -\alpha E_t + \frac{c_t}{\beta}, \quad (2)$$

where  $\alpha$  is the absorption ratio of CO<sub>2</sub>, and  $\beta$  denotes the efficiency of production measured by the amount of goods produced by per emitted CO<sub>2</sub> tonnage.

The maximization problem can be formatted as

$$\max_{c_t} \int_0^{+\infty} u(t) dt, \quad \text{subject to (2)}. \quad (3)$$

The corresponding Hamiltonian,  $H_t^U$ , of this problem is defined as

$$H_t^U \equiv u(c_t, -E_t) e^{-\rho t} + \lambda_t^U \left[ -\alpha [-E_t] - \frac{c_t}{\beta} \right]. \quad (4)$$

It should be noted that the state variable,  $E_t$ , is interchanged with  $-E_t$ . The necessary and sufficient conditions of this problem are Equation (2) and

$$\begin{aligned} u_1 &= \frac{\tilde{\lambda}_t^U}{\beta}, \tilde{\lambda}_t^U \equiv \lambda_t^U e^{\rho t}, \\ \dot{\tilde{\lambda}}_t^U &= [\alpha + \rho] \tilde{\lambda}_t^U - u_2, \\ \lim_{t \rightarrow +\infty} \tilde{\lambda}_t^U E_t e^{-\rho t} &= 0. \end{aligned} \quad (5)$$

The optimal paths are illustrated by bold arrows in Figure 1. In a situation with excess emissions situation, which corresponds to the lower arrow directing to north-east, optimal consumption is monotonously decreasing and converges to a zero-growth economy unless some technological progress, which absorbs emissions, is developed.

Next, we analyze the properties of the optimal social discount rate in the vicinity of the stationary state. Combining the above two equations in (5) and letting  $\dot{\tilde{\lambda}}^U = 0$ , one

obtains the following important relationship between optimal consumption and emissions. That is,

$$\frac{u_1}{u_2} = \frac{1}{\beta[\alpha + \rho]}. \quad (6)$$

The left-hand side of Equation (6) corresponds to the tangency of an upward-sloping indifference curve  $II$  on  $(c, E)$  plane as illustrated by Figure 2. By Equation (2),  $E = \frac{c}{\alpha\beta}$  holds as indicated by Line  $FF$ , which represents the feasible allocation in the stationary equilibrium; thus the stationary state is located at the intersection of curve  $II$  and  $FF$ . Because the indifference curve  $I^0I^0$ , which corresponds to zero social discount rate achieves the best allocation, it is evident that nearly-zero discounting (note that the utility integral diverges when zero-discounting is strictly applied) improves economic welfare in the vicinity of the stationary state. To summarize,

**Theorem 1**

*Welfare is maximized when the social discount rate is preferable as low as possible in the vicinity of stationary state in the utility function approach.*

Moreover, additional important information exists in Equation (6). Let a market economy be considered instead of a planned economy. Whenever perfect competition prevails in goods market, the tangency of the indifference curve is equal to the relative price of carbon to consumption goods. Hence, Equation (6) represents the effective carbon price (i.e., tax) in terms of consumption goods. Such a price becomes equal to the capitalized value of goods discounted by the *effective* social discount rate,  $\beta[\alpha + \rho]$ <sup>2</sup>. A society that becomes more tolerant to emissions and more highly discounts future utilities, whilst possibly not, would imply a lower carbon tax in a market economy.

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<sup>2</sup> From the equations in (5), the carbon price becomes equal to

$$\frac{1}{\beta \left[ [\alpha + \rho] - \frac{\dot{\tilde{\lambda}}^U}{\tilde{\lambda}^U} \right]}.$$

Meanwhile it is evident from Figure 1 that  $\frac{\dot{\tilde{\lambda}}^U}{\tilde{\lambda}^U}$  is strictly positive when an economy is in the situation of excess emissions compared with the stationary state. Accordingly, the optimal carbon price in transition process should be always higher than in the stationary state. In other words, the desired carbon price becomes minimal at the stationary state.

Thus, one obtains

**Theorem 2**

*A higher social discount rate in a planned economy is equivalent to a lower carbon tax in a market economy.*

It should be noted that the Earth’s absorption rate,  $\alpha$ , is an important factor for determining the carbon price. A large value of  $\alpha$  permits a lower carbon price because emitted CO<sub>2</sub> does not stay long in the atmosphere. Conversely, if  $\alpha$  does not significantly differ from zero, the solution to global warming depends on human beings’ ethic, as Otaki (2015) proves.

In addition, technological progress is also meaningful in introducing a carbon tax with less friction. Whenever some emissions-saving technology is developed and prevails, this is represented by heightening parameter  $\beta$  in the model. As indicated in Equation (6), such a progress lowers the relative price of carbon because the production process produces fewer emissions.

**3. Production Function Approach**

Under a production function approach, the negative effect of excess emissions is attributed to the production function rather than to the utility function directly. This approach assumes that global warming does not affect the basic living and survival conditions of human beings, but more indirectly, hinders prosperity via lowering productivity.

Let the strictly concave production function,  $F$ , be denoted as

$$y_t = F(k_t, -E_t), F_1 > 0, F_2 > 0, \tag{7}$$

where  $y_t$  is total output of goods, and  $k_t$  denotes input of goods for production. CO<sub>2</sub> emissions originating from anthropologic combustion of fossil fuels lowers productivity; thus, the second argument of  $F$  is positive because of the negative externality to production. Then, the differential equation that represents the carbon cycle becomes

$$\dot{E}_t = -\alpha E_t + \frac{y_t}{\beta} = -\alpha E_t + \frac{1}{\beta} F(k_t, -E_t). \tag{8}$$

The optimization problem to be solved is

$$\max_{c_t} \int_0^{+\infty} v(c_t) e^{-\rho t} dt, \text{ subject to Eqs. (8) and (9),} \tag{9}$$

where  $v$  is a strictly concave instantaneous utility function. The corresponding Hamiltonian,  $H^P$ , is

$$H^P \equiv v(F(k_t, -E_t) - k_t) e^{-\rho t} + \lambda_t^P \left[ -\alpha[-E_t] - \frac{1}{\beta} F(k_t, -E_t) \right]. \quad (10)$$

The necessary-sufficient conditions are Equation (8) and

$$\begin{aligned} v' &= \frac{\tilde{\lambda}_t^P}{\beta} \frac{F_1}{[F_1 - 1]}, \quad \tilde{\lambda}_t^P \equiv \lambda_t^P e^{\rho t}, \\ \dot{\tilde{\lambda}}_t^P &= \left[ \alpha + \rho + \frac{F_2}{\beta} \right] \tilde{\lambda}_t^P - v' F_2, \\ \lim_{t \rightarrow +\infty} \tilde{\lambda}_t^P E_t e^{-\rho t} &= 0. \end{aligned} \quad (11)$$

The dynamics of this economy are described by two differential equations (8) and (11). Figure 3 is its phase diagram with the optimal paths shown by bold arrows.

By evaluating Equation (8) and the top two equations in (11) at the stationary state, one obtains

$$\frac{dE}{dc} = \frac{F_1 - 1}{F_2} = \frac{1}{\beta[\alpha + \rho]}. \quad (12)$$

The left-hand side of Equation (12) is the slope of an isoquant. Figure 4 illustrates stationary state in production function approach model. Curve  $F^P F^P$  is the feasible set, which corresponds to Equation (8) evaluated at  $\dot{\tilde{\lambda}}^P = 0$ . Curves  $IQ, IQ^O$  are isoquants in  $(c, E)$  plane. The same logic as in the case of the utility-function approach can be applied; thus, the relative price of carbon becomes equal to the tangency of isoquant,  $\frac{1}{\beta[\alpha + \rho]}$ , in the production-function approach. This is because the tangency corresponds to the factor price whenever perfect competition prevails. Accordingly,

### Theorem 3

*Regardless of whether one relies on the utility-function or production-function approach, the optimal carbon in stationary state is equal to  $\frac{1}{\beta[\alpha + \rho]}$ . In addition, the relationship between the carbon price and the social discount rate is invariant.*

Theorem 3 implies that there is no significant difference in the calculation methods of

the carbon price in the vicinity of a stationary equilibrium. In this sense, whether one adopts a utility-function or production-function approach is immaterial.

#### **4. Concluding Remarks**

This study considered the welfare implications of the social discount rate and calculated the optimal carbon price using an optimal growth model with an intertemporal negative externality. In addition, it examined whether results differ when the negative externality exists in the utility or the production function. The results obtained are as follows:

First, the optimal carbon price in the vicinity of stationary state is independent of the origin of the negative externality: be it either the utility function or the production function. In addition, the adjustment processes toward a stationary state are alike: consumption monotonously decreases and the concentration of CO<sub>2</sub> is gradually reduced.

Second, as the social discount rate approaches zero, resource allocation in the vicinity of stationary equilibrium improves. Nevertheless, applying a lower discount rate raises the carbon price in a market economy.

Third, the absorption ratio of CO<sub>2</sub> by the Earth is also a crucial factor in determining the carbon price. If this ratio is high, the carbon price is lower because emitted CO<sub>2</sub> does not remain in the atmosphere for a long time. However, it should be noted that the acidification of the oceans may not lead a stable equilibrium in the long run.

Finally, increased carbon-efficiency of production lowers the carbon price. This is because such a technological progress enables an economy to produce more goods in exchange for lower emission. As Kuninori and Otaki (2016) shows, it is only when such technological progress continuously accumulates that sustainable growth is feasible.

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## Appendix: Derivation of Equation (12)

From the top and middle equations in (11) evaluated at  $\tilde{\lambda}_t^P = 0$ , one obtains

$$\begin{aligned}v' F_2 &= \frac{v'[F_1 - 1]}{F_1} [\beta[\rho + \alpha] + F_2] \\ \Rightarrow F_2 &= \frac{[F_1 - 1]}{F_1} [\beta[\rho + \alpha] + F_2] \\ \Rightarrow \frac{[F_1 - 1]}{F_2} &= \frac{1}{\beta[\rho + \alpha]}.\end{aligned}$$

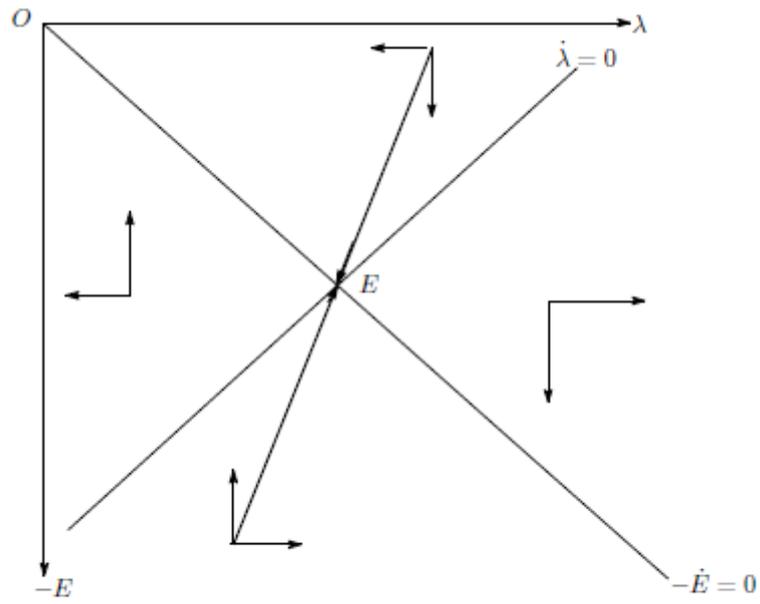


Figure 1: The Phase Diagram

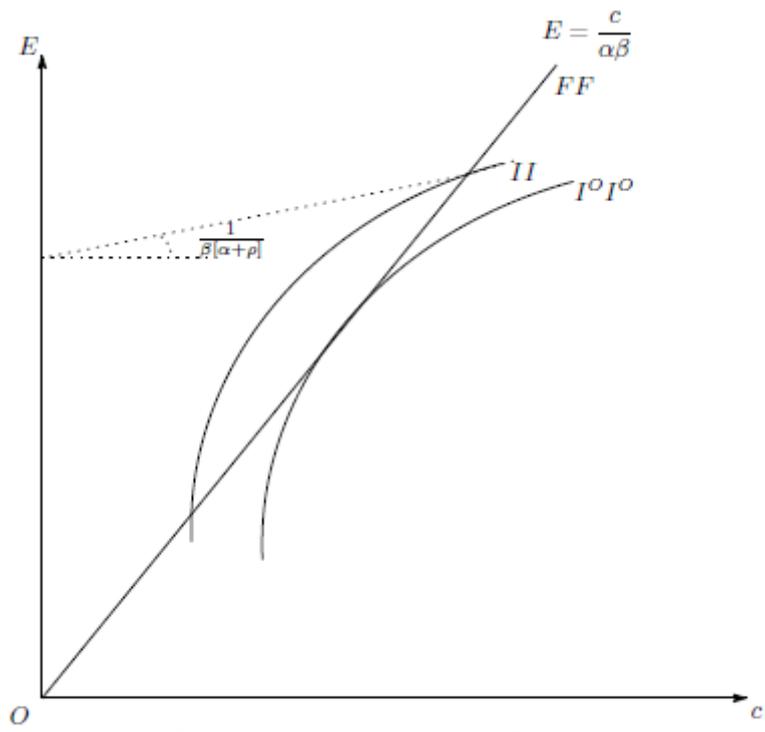


Figure 2: Stationary State and Pareto Efficiency

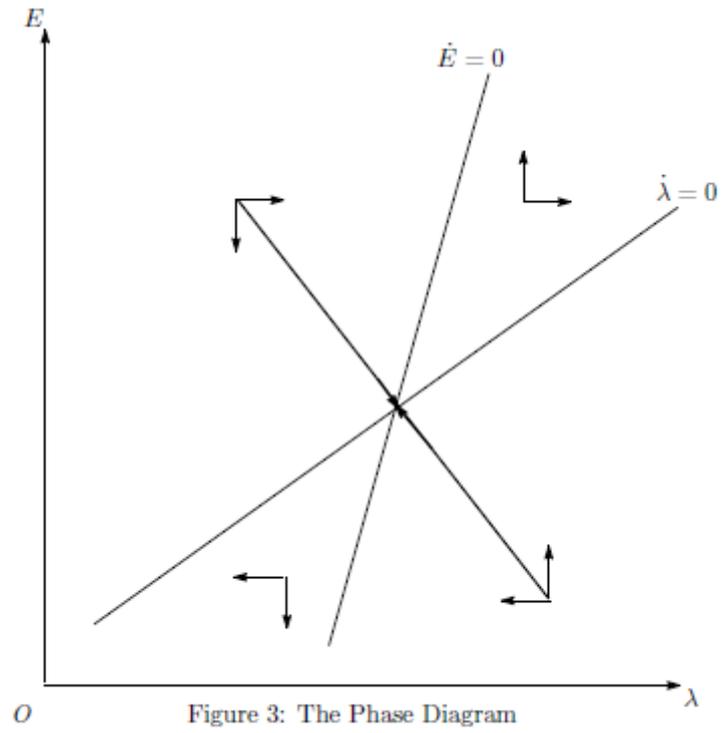


Figure 3: The Phase Diagram

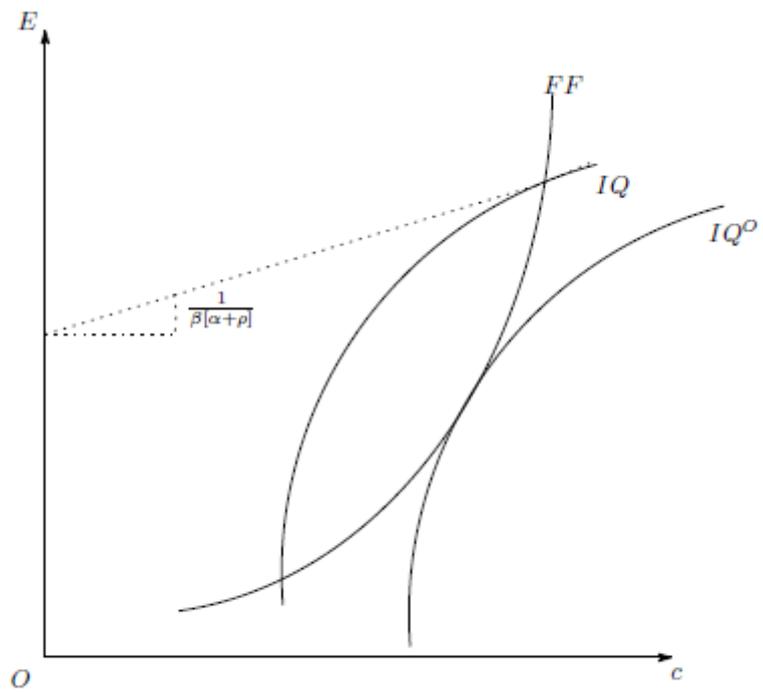


Figure 4: Stationary State and Carbon Price