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A Study
on
Lucas' ``Expectations and the Neutrality of Money´´

Masayuki Otaki
(Institute of Social Science, University of Tokyo)

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Institute of Social Science, University of Tokyo

Abstract

This short article shows that the functional equation on the equilibrium price function is more complicated than that considered by Lucas [1], and that modification is required to complete the proof. Furthermore, we provide a sufficient condition that guarantees the uniqueness of the equilibrium price function.

Key words: Neutrality of Money,
Functional Equation, Contraction Mapping
A Sufficient Condition for the Unique Equilibrium Price Function
JEL Classification: E31, E41

1 Introduction

This study aims to show that an additional condition is necessary for the operator T in Lucas [1] to become the contraction mapping. This is because transformation between the functional equations on the equilibrium price function is not equivalent. We also provide a sufficient condition such that T can become the contraction mapping.

This paper proceeds as follows. Section 2 rewrites the functional equation in the correct form and shows that, contrary to the implication of the original paper, the contraction mapping method cannot be easily applied. A sufficient condition for the uniqueness of the equilibrium price function is provided in Section 3. Section 4 contains brief concluding remarks.

2 Equivalent Transformation

Lucas [2] admits that there is no guarantee that p_0 and z exhibit a one-to-one correspondence, and some reservation is necessary for the conclusion. He also recognizes that the equilibrium price function $p(m, x, \theta)$ should be specified as $m\phi(\frac{x}{\theta})$ to determine the unique equilibrium. Besides these problems, the transformation between functional equations below is not equivalent. The present study aims to clarify this fact and show the rather restrictive condition for supporting the original result.

Lucas [1] first derives the following functional equation as the equilibrium condition of the money market:

$$h[\frac{x}{\theta\phi(\frac{x}{\theta})}]\frac{x}{\theta\phi(\frac{x}{\theta})} = \int V'[\frac{xx'}{\theta\xi\phi(\frac{x'}{\theta'})}]\frac{xx'}{\theta\xi\phi(\frac{x'}{\theta'})}dG(\xi, x', \theta' \mid \frac{x}{\theta}), \tag{1}$$

where x is the realized value of the increment of money during the current period. θ denotes the realize value of the population of the young generation. x', θ' are random variables of each exogenous shock during the subsequent period. We must note the existence of the random variable ξ. Although $z \equiv \frac{x}{\theta}$ is available information via the inverse equilibrium price function, x cannot be directly observed by household. Thus, when x singly appears in the functional equation, it should be treated as the random variable ξ.

The right-hand side of (1) represents the marginal utility of the current consumption, and the left-hand side represents the expected marginal utility of the future consumption. Namely, functional equation (1) is the Euler equation in this model. Lucas [1] asserts that (1) is equivalently transformed into

$$h[\frac{z}{\phi(z)}]\frac{z}{\phi(z)} = \int V'[\frac{\theta'}{\theta}\frac{z'}{\phi(z')}]\frac{\theta'}{\theta}\frac{z'}{\phi(z')}\tilde{H}(z,\theta)H(z',\theta')d\theta dz' d\theta'. \tag{2}$$

However, (1) and (2) are not equivalent. We shall address this problem. This transformation assumes that $\xi = x$. Nevertheless, as discussed above, x is a realized vale (real number) of the random variable ξ (measurable function). Hence, they cannot be cancelled out. The equivalent transformation from (1) to (2) is

$$h[\frac{z}{\phi(z)}]\frac{z}{\phi(z)} = \int V'[\frac{z\theta'}{\xi}\frac{z'}{\phi(z')}]\frac{z\theta'}{\xi}\frac{z'}{\phi(z')}dG(\xi, x', \theta' \mid z). \tag{3}$$

Let us define $\Psi(z) \equiv h[\frac{z}{\phi(z)}]\frac{z}{\phi(z)}$, $G^{-1}(x) \equiv xh(x)$ and $G_2(x) \equiv V'(x)x$. Using these definitions, (1) is transformed into

$$\Psi(z) = \int G_2[\frac{z\theta'}{\xi}G_1(\Psi(z'))]dG(\xi, x', \theta' \mid z).$$

Then the correct form of the operator T in the Appendix of Lucas [1] becomes

$$Tf \equiv \ln \int G_2[\frac{z\theta'}{\xi}G_1(e^{f(z')})]dG(\xi, x', \theta' \mid z). \tag{4}$$

Consequently, inequality (A.6) in Lucas' [1] appendix

$$\| Tf - Tg \| \leq \sup_{z,\theta',\theta} | \ln G_2[\frac{\theta'}{\theta}G_1(e^{f(z)})] - \frac{\theta'}{\theta}\ln G_2[G_1(e^{g(z')})] | \quad (A.6)$$

is modified as

$$\| Tf - Tg \| \le \sup_{z} \sup_{z',\theta',\xi} |\ln G_2[\frac{z\theta'}{\xi} G_1(e^{f(z')})] - \ln G_2[\frac{z\theta'}{\xi} G_1(e^{g(z')})]|. \tag{5}$$

It is noteworthy that z', θ', and ξ are functions of z in (5). Let us denote these functions as

$$z' = \omega(z), \quad \frac{\theta'}{\xi} = \chi(z). \tag{6}$$

Accordingly, (5) becomes

$$\| Tf - Tg \| \le \sup_{z} |\ln G_2[z\chi(z) G_1(e^{f(\omega(z))})] - \ln G_2[z\chi(z) G_1(e^{g(\omega(z))})]|. \tag{7}$$

Let us define z^*, x_1, and x_2 as

$$z^* \equiv \arg\sup_{z} |\ln G_2[z\chi(z) G_1(e^{f(\omega(z))})] - \ln G_2[z\chi(z) G_1(e^{g(\omega(z))})]|.$$

$$x_1 \equiv f(\omega(z^*)), \quad x_2 \equiv g(\omega(z^*)).$$

Consequently, (7) is transformed into

$$\| Tf - Tg \| \le |\ln G_2[z^* \chi(z^*) G_1(e^{x_1})] - \ln G_2[z^* \chi(z^*) G_1(e^{x_2}))]|. \tag{8}$$

Applying the mean value theorem and Lucas' [1] assumptions (A.2) and (A.3)

$$0 < \frac{x G_{1'}(x)}{G_1(x)} < 1 \ (A.2), \quad 0 < \frac{x G_{2'}(x)}{G_2(x)} \le 1 - a < 1 \ (A.3)$$

to (8), we finally obtain

$$\| Tf - Tg \| \le (1-a) | x_1 - x_2 | \le (1-a) \| f(\omega(z)) - g(\omega(z)) \|. \tag{9}$$

Since $\omega(z) \ne z$ generally, the original paper has not succeeded in proving that the operator T is the contraction mapping.[1] Thus, an additional condition is necessary to complete the proof.

3 A sufficient condition

Since the difficulty arises from the fact that (5) explicitly depends on z, we assume that the function G_2 is multiplicatively separable. Namely, suppose that G_2 satisfies

$$G_2(xy) = G_2(x) G_2(y). \tag{10}$$

In this case, (5) is modified as

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[1] It is noteworthy that

$$\| f(\omega(z)) - g(\omega(z)) \| \le \| f(z) - g(z) \|.$$

does not necessarily hold. For example, when $\omega(z) = 2z$, $g < f$, and f is a strictly increasing function, g is strictly decreasing, then

$$\| f(2z) - g(2z) \| \ge \| f(z) - g(z) \|.$$

holds.

$$\| Tf - Tg \| \leq \sup_{z',\theta',\xi} | \ln G_2[\frac{\theta'}{\xi} G_1(e^{f(z')})] - \ln G_2[\frac{\theta'}{\xi} G_1(e^{g(z')})] |. \quad (11)$$

This inequality is essentially identical to (A.6), and thus, T becomes the contraction mapping.

Nevertheless, the function G_2, which satisfies the functional equation (10), is confined to power functions (see Small [3]). Hence,

$$V'(x)x = x^{\beta} \quad \Rightarrow \quad V(x) = \frac{x^{\beta}}{\beta}. \quad (12)$$

(A.3) also requires that $0 < \beta < 1$. In sum, the CRRA (Constant Relative Risk Aversion) family, whose relative risk aversion is located within $(0,1)$, is the only function satisfying the sufficient condition (10).

4 Conclusion

We have shown that the functional equation of the equilibrium price function is more complicated than that considered by Lucas [1]. Hence, some additional condition is necessary to employ the contraction mapping method. This study finds that if V belongs to CRRA family of low relative risk aversion, the uniqueness of the solution is guaranteed. In sum, Lucas' assertion on the neutrality of money under uncertainty holds only under certain utility functions that are rather restrictive as compared to those previously considered.

References

[1] R. E. Lucas, Jr., ''Expectations and the Neutrality of Money,'' *Journal of Economic Theory*, Vol. 4, 1972, pp. 103-124.

[2] R. E. Lucas, Jr., ''Corrigendum to `Expectations and the Neutrality of Money,'' *Journal of Economic Theory*, Vol. 31, 1983, pp. 197-199.

[3] C.G. Small, Functional Equations and How to Solve Them, Springer Science+Business Media, New York 2007.