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Social Common Capital, Imputed Price, and Sustainable Development

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In this paper, we prove in terms of the prototype model of social common capital that the optimum conditions for sustainable processes of capital accumulation involving both private capital and social common capital coincide precisely with those for market equilibrium with the social common capital taxes at certain specific rates under the stationary expectations hypothesis concerning the future schedule of marginal productivity of capital of all kinds.

Keywords: Social common capital, Prototype model of social common capital, Sustainable processes of capital accumulation, Social common capital tax Imputed price, Stationary expectations hypothesis

1. Introduction

Social common capital involves intergenerational equity and justice. Although the construction and maintenance of social common capital require the use of substantial portions of scarce resources, both human and non-human, putting a significant burden on the current generation, but the people in future generations will benefit greatly if the construction of social common capital carried out by the current generation is properly arranged.

In this paper, we examine the problems of the accumulation of social common capital primarily from the viewpoint of the intergenerational distribution of utility. Our analysis is based on the concept of sustainability introduced in Uzawa (2003, 2005), and we examine the conditions under which processes of the accumulation of social common capital over time are sustainable. The conceptual framework of the economic analysis of social common capital developed in Uzawa (2005) is extended to deal with the problems of the irreversibility of processes of the accumulation of social common capital due to the Penrose effect. The concept of the Penrose effect was originally introduced in Uzawa (1968, 1969) in the context of macro-economic analysis, and was extensively utilized in the dynamic analysis of global warming as in detail described in Uzawa (2003, 2005). The presentation of the theory of sustainable processes of capital accumulation in this paper largely reproduces the one introduced there.

The existence of the sustainable time-path of consumption and capital accumulation starting with an arbitrarily given stock of capital is ensured when the processes of accumulation of various kinds of capital are subject to the Penrose effect that exhibits the law of diminishing marginal rates of investment.

In what follows, we formulate the concept of sustainability within the theoretical framework of the economic analysis of social common capital in such a manner that it may be consulted in devising institutional arrangements and policy measures in realizing the stationary state in the sense introduced by John Stuart Mill in his classic *Principles of Political Economy* (Mill, 1848), particularly in the chapter entitled "Of the Stationary State." The stationary state, as envisioned by Mill, is interpreted as the state of the economy in which all macro-economic variables, such as gross domestic product, national income, consumption, investments, wages, and real rates of interest, remain stationary, whereas, within the society, individuals are actively engaged in economic, social, and cultural activities, new scientific discoveries are incessantly made, and new products are continuously introduced while the natural environment is being preserved at the sustainable state.

Our analysis is focused upon the role of the imputed prices of the stock of various components of capital, both private capital and social common capital, in the processes of sustainable economic development. The imputed price ψ_t of each kind of capital at a particular time t expresses the extent to which the marginal increase in the stock of that kind of capital at time t induces the marginal increase in units of market prices in the welfare level of each country in units of market prices, including those of all future generations. The imputed price ψ_t is at the sustainable level at time t, if it remains stationary at time t; i. e.,

$$\dot{\psi}_t = 0 \quad \text{at time } t,$$

where $\dot{\psi}_t$ refers to the time derivative with respect to the time of the virtual capital market at time t. A time-path of the accumulations of each kind of capital is defined sustainable, if the imputed price ψ_t is at the sustainable level at all times t.

Because the sustainability of the imputed price is defined with respect to the fictitious time of the virtual capital market at time t, the sustainability of the imputed price does not necessarily imply the stationarity of the imputed price, as inadvertently stated in Uzawa (2003, 2005). All the policy and institutional conclusions obtained there, however, remain valid with regard to the concept of the sustainability of time-paths of the processes of capital accumulation, both private capital and social common capital, as introduced in this paper.

2. Sustainability in the Aggregative Model of Capital Accumulation

The basic premises of the analysis of sustainability are that the intertemporal preference ordering prevailing in the society in question is independent of the technological conditions and processes of capital accumulation.

We consider an aggregative model of capital accumulation whose behavioral characteristics are described by those of the representative consumer and producer. We first consider the simple case in which only one kind of goods serves both for consumption and investment.

The instantaneous level of the utility u_t at each time t is represented by a utility function

$$u_t = u(c_t),$$

where c_t is the quantity of goods consumed by the representative consumer at time t.

We assume that the utility function $u = u(c)$ is defined for all nonnegative $c \geqq 0$, is continuous and continuously twice-differentiable, and satisfies the following conditions:

$$u(c) > 0, \quad u'(c) > 0, \quad u''(c) < 0 \quad \text{for all} \quad c > 0.$$

We denote by K_t the stock of capital at time t, and by c_t, z_t, respectively, consumption and investment at time t. Then, we have

$$c_t + z_t = f(K_t), \quad c_t, \quad z_t \geqq 0. \qquad (1)$$

where $f(K)$ is the production function.

The production function $f(K)$ expresses the gross national product produced from the given stock of capital K. We assume that production function $f(K)$ is defined, continuous, and continuously twice-differentiable for all $K \geqq 0$, that marginal product is always positive and production processes are subject to the law of diminishing marginal returns:

$$f(K) > 0, \quad f'(K) > 0, \quad f''(K) < 0 \quad \text{for all} \quad K > 0.$$

The rate of capital accumulation at time t, $\dot{K}_t$, is given by the differential equation

$$\dot{K}_t = \alpha(z_t, K_t) - \mu K_t, \quad K_0 = K_o \tag{2}$$

where $\alpha(z_t, K_t)$ is the Penrose function relating the rate of capital accumulation $\dot{K}_t$ to investment z_t and the stock of capital K_t at time t, K_o is the initial stock of capital, and μ is the rate of depreciation.

The Penrose function $\alpha(z, K)$ expresses the gross rate of capital accumulation; thus we may assume that the partial derivative of $\alpha(z, K)$ with respect to z is always positive, whereas with respect K it is always negative:

$$\alpha = \alpha(z, K) \geqq 0, \quad \alpha_z = \alpha_z(z, K) > 0, \quad \alpha_K = \alpha_K(z, K) < 0.$$

The *Penrose effect* is expressed by the conditions that the Penrose function $\alpha(z, K)$ is concave and strictly quasi-concave with respect to (z, K):

$$\alpha_{zz} < 0, \quad \alpha_{KK} < 0, \quad \alpha_{zz}\alpha_{KK} - \alpha_{zK}^2 \geqq 0 \quad \text{for all} \quad z \geqq 0, \quad K \geqq 0.$$

The following condition is usually assumed for the Penrose function $\alpha(z, K)$:

$$\alpha_{zK} = \alpha_{Kz} < 0.$$

The concept of the Penrose effect was originally introduced by Penrose (1959) to describe the growth processes of an individual firm. It was later formalized by Uzawa (1968, 1969) in the context of a Keynesian analysis of macro-economic processes of dynamic equilibrium to elucidate the effect of investment activities on the processes of capital accumulation.

2.1. Marginal Efficiency of Investment

A particularly important concept associated with the Penrose function $\alpha(z, K)$ is *marginal efficiency of investment*, which plays a crucial role in the analysis of dynamic processes of capital accumulation and economic growth. Marginal efficiency of investment expresses the extent to which the marginal increase in investment z induces the marginal increase in gross national product $f(K)$ in the future. The marginal efficiency of investment is composed of two components.

The first component is the marginal increase in gross national product $f(K)$ directly induced by the marginal increase $\alpha_z(z, K)$ in the stock of capital due to the marginal increase in investment z; that is, $r(K)\alpha_z(z, K)$, where $r(K) = f'(K)$ is the marginal product of capital.

The second component measures the extent of the marginal effect on future processes of capital accumulation due to the marginal increase in the stock of capital today, K; that is, $\alpha_K(z, K)$.

Thus, the marginal efficiency of investment $m = m(z, K)$ may be expressed as

$$m(z, K) = r(K)\alpha_z(z, K) + \alpha_K(z, K).$$

The marginal efficiency of investment $m(z, K)$ is a decreasing function of both investment z and the stock of capital K:

$$m_z(z, K) = \frac{\partial m}{\partial z} = r\alpha_{zz} + \alpha_{Kz} < 0$$

$$m_K(z, K) = \frac{\partial m}{\partial K} = r'\alpha_z + (r\alpha_{zK} + \alpha_{KK}) < 0,$$

where $r' = r'(K) = f''(K) < 0$.

In the standard neoclassical theory of investment, the Penrose effect is not recognized; that is,

$$\alpha(z, K) = z \quad \text{for all} \quad z \geqq 0, \quad K > 0.$$

Then,

$$m(z, K) = r(K)$$

$$m_z(z, K) = 0, \; m_K(z, K) = r'(K) < 0 \quad \text{for all} \quad z \geqq 0, \quad K > 0.$$

2.2. Imputed Price of Capital and Sustainability

The imputed price of capital at time t, ψ, is the discounted present value of the marginal increases in outputs in the future measured in units of the utility due to the marginal increase in the stock of capital at time t. The marginal increase in outputs at future time τ in units of the utility is given by $\psi_t m_\tau$, where m_τ is the marginal efficiency of investment at future time τ:

$$m_\tau = m(z_\tau, K_\tau) = r_\tau \alpha_z(z_\tau, K_\tau) + \alpha_K(z_\tau, K_\tau), \; r_\tau = f'(K_\tau). \qquad (3)$$

Thus the imputed price of capital at time t, ψ_t, is given by

$$\psi_t = \int_t^\infty m_\tau \psi_\tau e^{-(\delta+\mu)(\tau-t)} d\tau. \qquad (4)$$

By differentiating both sides of (4) with respect to time t, we obtain the following differential equation:

$$\dot{\psi}_t = (\delta + \mu)\psi_t - m_t \psi_t. \qquad (5)$$

Differential equation (5) is nothing but the Euler-Lagrange differential equation in the calculus of variations. In the context of the theory of optimum capital accumulation, it is often referred to as the Ramsey-Keynes equation. The economic meaning of the Ramsey-Keynes equation (5) may be brought out better if we rewrite it as

$$\dot{\psi}_t + m_t \psi_t = (\delta + \mu)\psi_t. \qquad (6)$$

We suppose that capital is transacted as an asset on the virtual capital market that is perfectly competitive and the imputed price ψ_t is identified with the market price at time t. Consider the situation in which the unit of such an asset is held for the short time period $[t, \; t+\Delta t] \; (\Delta t > 0)$. The gains obtained by holding such an asset are composed of "capital gains" $\Delta\psi_t = \psi_{t+\Delta t} - \psi_t$ and "earnings" $m_t \psi_t \Delta t$; that is,

$$\Delta\psi_t + m_t \psi_t \Delta t.$$

On other hand, the cost of holding such an asset for the time period $[t, \; t+\Delta t]$ consists of "interest payment" $\delta\psi_t \Delta t$ and "depreciation charges" $\mu\psi_t \Delta t$, where the social rate of discount δ is identified with the market rate of interest. Hence, on the virtual capital market, these two amounts become equal; that is,

$$\Delta\psi_t + m_t \psi_t \Delta t = (\delta + \mu)\psi_t \Delta t.$$

By dividing both sides of this equation by Δt and taking the limit as $\Delta t \to 0$, we obtain relation (6).

The imputed real national income at time t, H_t, is given by

$$H_t = u(x_t) + \psi_t[\alpha(z_t, K_t) - \mu K_t]. \qquad (7)$$

The optimum levels of consumption and investment at time t, (c_t, z_t), are obtained if imputed real national income H_t at time t is maximized subject to the feasibility constraints (1). Let the Lagrangian form be given by

$$L_t = u(c_t) + \psi_t[\alpha(z_t, K_t) - \mu K_t] + p_t[f(K_t) - c_t - z_t],$$

Where p_t is the Lagrangian unknown associated with constraints (1).

The optimum conditions are

$$u'(c_t) = p_t \qquad (8)$$

$$\psi_t \alpha_z(z_t, K_t) = p_t, \qquad (9)$$

where the value of p_t is chosen so that feasibility condition (1) is satisfied.

Lagrange unknown p_t may be interpreted as the imputed price of the output at time t. Equation (8) means that the optimum level of consumption c_t at time t is obtained when marginal utility $u'(c_t)$ is equated with imputed price at time t, p_t. Equation (9) means that the optimum level of investment z_t at time t is obtained when the value of the marginal product of investment z_t evaluated at the imputed price of capital ψ_t is equated with the imputed price of the output at time t, p_t.

The imputed price ψ_t is defined to be at the sustainable level at time t, if it remains stationary at time t; i. e.,

$$\dot{\psi}_t = 0 \quad \text{at time } t,$$

where $\dot{\psi}_t$ refers to the time derivative with respect to the time of the virtual capital market at time t. From the basic differential equation (6), the imputed price π_t is at the sustainable level at time t, if, and only if, the marginal efficiency of investment is equal to the sum of the social rate of discount and the rate of depreciation; i. e.,

$$m(z_t, K_t) = \delta + \mu \quad \text{at time } t. \qquad (10)$$

A time-path of capital accumulation is defined *sustainable*, if it is at the sustainable level at all times t.

2.3. Sustainable Processes of Capital Accumulation

First, we would like to see if the levels of consumption and investment at the sustainable time-path $[(c_t, z_t)]$ are uniquely determined. To see this, the conditions for sustainability are put together as follows:

$$\dot{K} = \alpha(z, K) - \mu K \tag{11}$$

$$c + z = f(K) \tag{12}$$

$$m(z, K) = r(K)\alpha_z(z, K) + \alpha_K(z, K) = \delta + \mu, \tag{13}$$

where the time suffix t is omitted.

By taking a differential of both sides of relations (12) and (13), we obtain

$$\begin{pmatrix} 1 & 1 \\ 0 & m_z \end{pmatrix}\begin{pmatrix} dc \\ dz \end{pmatrix} = \begin{pmatrix} r & 0 \\ -m_K & 1 \end{pmatrix}\begin{pmatrix} dK \\ d\delta \end{pmatrix},$$

where

$$m_z = r\alpha_{zz} + \alpha_{Kz} < 0, \;\; m_K = r'\alpha_z + (r\alpha_{zK} + \alpha_{KK}) < 0 \quad [r' = f''(K) < 0].$$

Hence,

$$\begin{pmatrix} dc \\ dz \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & m_z \end{pmatrix}^{-1}\begin{pmatrix} r & 0 \\ -m_K & 1 \end{pmatrix}\begin{pmatrix} dK \\ d\delta \end{pmatrix} = \frac{1}{m_z}\begin{pmatrix} rm_z + m_K & -1 \\ -m_K & 1 \end{pmatrix}\begin{pmatrix} dK \\ d\delta \end{pmatrix}$$

$$\frac{\partial c}{\partial K} = r + \frac{m_K}{m_z} > 0, \;\; \frac{\partial z}{\partial K} = -\frac{m_K}{m_z} < 0, \;\; \frac{\partial c}{\partial \delta} = -\frac{1}{m_z} > 0, \;\; \frac{\partial z}{\partial \delta} = \frac{1}{m_z} < 0.$$

Thus, the levels of consumption and investment (c, z) at the sustainable time-path are uniquely determined. In addition, we have

$$\frac{\partial \alpha}{\partial K}\Big|_{\dot{K}=0} = \alpha_z \frac{\partial z}{\partial K} + \alpha_K - \mu = -\alpha_z \frac{m_K}{m_z} + \alpha_K - \mu < 0, \;\; [\alpha_z > 0, \;\; \alpha_K < 0].$$

Hence, the differential equation (11) has a uniquely determined stationary state, and it is globally stable, approaching the long-run stationary state, identical with that of the dynamically optimum level.

Thus, we have established the following proposition.

Proposition 1. *For an economy with only one kind of capital, the sustainable processes of capital accumulation are obtained if, and only if, the marginal efficiency of investment is equal to the sum of the social rate of discount and the rate of depreciation at all times* t; *i.e.,*

$$m(z_t, K_t) = \delta + \mu \quad \textit{at all times } t.$$

The levels (c_t, z_t) *of consumption and investment along the sustainable tame-path are uniquely determined for any given stock of capital* $K_o > 0$.

Along the sustainable time-path, the larger the stock of private capital K_t, *the higher is the level of consumption* c_t *and the lower is the level of investment* z_t. *The higher the rate of discount* δ_t, *the higher is the level of consumption* c_t *and the lower is the level of investment* z_t.

At the sustainable time-path, the stock of capital K_t approaches, as time t *goes to infinity, the long-run stationary state* K^* *that is dynamically optimum.*

For the standard case of the neoclassical world, we have

$$\alpha(z, K) = z$$

$$m(z, K) = f'(K) \quad \text{for all } z, \ K \geqq 0.$$

Hence, a z that satisfies sustainability conditions (13) does not generally exist.

3. The Prototype Model of Social Common Capital

The analysis of sustainable processes of consumption and capital accumulation we have introduced in the previous section may be readily applied to the prototype model of social common capital as introduced in Uzawa (2005).

In the prototype model of social common capital introduced in Uzawa (2005), we consider a particular type of social common capital --- social infrastructure, such as public utilities, public transportation systems, ports, and highways. We consider the general circumstances where factors of production that are necessary for the professional provision of services of social common capital are either privately owned or managed as if private owned. Services of social common capital are subject to the phenomenon of congestion, resulting in the divergence between private and social costs. Therefore, to obtain efficient and equitable allocation of scarce resources, it becomes necessary to levy taxes on the use of services of social common capital. The prices charged for the use of services of social common capital exceed, by the tax rates, the prices paid to social institutions in charge of the provision of services of social common capital. One of crucial problems in the economic analysis of social common capital is to examine how the optimum tax rates for the services of various components of social common capital are determined. The nature of services of social common capital varies to such a significant degree that it is extremely difficult to formulate a unifying theory concerning the determination of the optimum taxes on services of social common capital. The prototype model of social common capital introduced in this paper incorporates some of the more salient features of social common capital, and the analytical apparatuses and institutional and policy implications regarding the prototype model of social common capital may serve as guidelines for the analysis of the specific types of social common capital.

3.1. Basic Premises of the Prototype Model of Social Common Capital

Although the basic premises of the model remain identical with those for the prototype model of social common capital in Uzawa (2005), we must explicitly take into account the investment activities in both private firms and social institutions in charge of social common capital. For the sake of expository brevity, we assume that only fixed factors of production are limitational in the production processes of both private firms and social institutions in charge of social common capital. The following analysis may be easily extended to the general case in which variable factors, such as labor and energy input, are essentially required in the production processes.

We consider an economy consisting of n individuals, m private firms, and s social institutions in charge of social common capital. Individuals are generically denoted by $\nu = 1, .., n$, private firms by $\rho = 1, .., m$, and social institutions by $\sigma = 1, .., s$. Goods produced by private firms are generically denoted by $j = 1, .., J$. Fixed factors of production are generically denoted by $f = 1, .., F$, whereas there is only one kind of social common capital.

3.2. Individuals

The utility of each individual ν is cardinal and is expressed by the utility function

$$u^{\nu} = u^{\nu}(c^{\nu}, \varphi^{\nu}(a) a^{\nu}),$$

where c^{ν} is the vector of goods consumed and a^{ν} is the amount of services of social common capital used, both by individual ν, whereas a is the total amount of services of social common capital used by all members of the society:

$$a = \sum_{\nu} a^{\nu} + \sum_{\rho} a^{\rho},$$

where a^{ρ} is the amount of services of social common capital used by private firm ρ. The impact index function $\varphi^{\nu}(a)$ expresses the extent to which the utility of individual ν is affected by the phenomenon of congestion with respect to the use of services of social common capital. The impact coefficients $\tau(a)$ of social common capital defined by

$$\tau(a) = -\frac{\varphi^{\nu\prime}(a)}{\varphi^{\nu}(a)} > 0$$

are assumed to be identical for all individuals and satisfies the following conditions:

$$\tau(a) > 0, \ \tau'(a) > 0.$$

The utility function $u^{\nu}(c^{\nu}, a^{\nu})$ is assumed to satisfy the following conditions:

(U1) $u^\nu(c^\nu,\ a^\nu)$ is defined, positive, continuous, and continuously twice-differentiable with respect to $(c^\nu,\ a^\nu)$ for all $(c^\nu,\ a^\nu) \geqq 0$.

(U2) $u^\nu_{c^\nu}(c^\nu,\ a^\nu) > 0,\ u^\nu_{a^\nu}(c^\nu,\ a^\nu) > 0$ for all $(c^\nu,\ a^\nu) \geqq 0$.

(U3) Marginal rates of substitution between any pair of consumption goods and services of social common capital are diminishing, or more specifically, $u^\nu(c^\nu,\ a^\nu)$ is strictly quasi-concave with respect to $(c^\nu,\ a^\nu)$.

(U4) $u^\nu(c^\nu,\ a^\nu)$ is homogeneous of order 1 with respect to $(c^\nu,\ a^\nu)$.

3.3. Private Firms

Processes of production in private firms are also affected by the phenomenon of congestion regarding the use of services of social common capital. We assume that, in each private firm ρ, the minimum quantities of factors of production that are required to produce goods by x^ρ and at the same time to increase the stock of fixed factors of production by $z^\rho = (z^\rho_f)$ with the use of services of social common capital at the level a^μ are specified by the following vector-valued function:

$$f^\rho(x^\rho, z^\rho, \varphi^\rho(a)a^\rho) = (f^\rho_f(x^\rho, z^\rho, \varphi^\rho(a)a^\rho)),$$

where $\varphi^\rho(a)$ is the impact index with regard to the extent to which the effectiveness of services of social common capital in processes of production in private firm ρ is impaired by congestion. For private firm ρ, the impact coefficients $\tau^\rho(a)$ of social common capital to be defined by

$$\tau^\rho(a) = -\frac{\varphi^{\rho}{}'(a)}{\varphi^\rho(a)}$$

are assumed to be identical for all private firms, identical to those for individuals ; i.e.,

$$\tau^\rho(a) = \tau(a) \quad \text{for all} \quad \rho,$$

The production possibility set of each private firm ρ, T^ρ, is composed of all combinations $(x^\rho, z^\rho,\ a^\rho)$ of vectors of production x^ρ and investment z^ρ, and use of services of social common capital a^ρ that are possible with the organizational arrangements, technological conditions, and given endowments of factors of production K^ρ in firm ρ. It may be expressed as

$$T^\rho = \left\{ (x^\rho, z^\rho, a^\rho) : (x^\rho, z^\rho, a^\rho) \geqq 0, \quad f^\rho(x^\rho, z^\rho, \varphi^\rho(a) a^\rho) \leqq K^\rho \right\},$$

where the total amount of services of social common capital used by all members of the society, a, is assumed to be a given parameter.

The following neoclassical conditions are assumed:

($T^\rho 1$) $f^\rho(x^\rho, z^\rho, a^\rho)$ are defined, positive, continuous, and continuously twice-differentiable with respect to (x^ρ, z^ρ, a^ρ).

($T^\rho 2$) $f^\rho_{x^\rho}(x^\rho, z^\rho, a^\rho) > 0, \; f^\rho_{z^\rho}(x^\rho, z^\rho, a^\rho) > 0, \; f^\rho_{a^\rho}(x^\rho, z^\rho, a^\rho) < 0$.

($T^\rho 3$) $f^\rho(x^\rho, z^\rho, a^\rho)$ are strictly quasi-convex with respect to (x^ρ, z^ρ, a^ρ).

($T^\rho 4$) $f^\rho(x^\rho, z^\rho, a^\rho)$ are homogeneous of order 1 with respect to $(x^\rho, \ell^\rho, a^\rho)$.

3.4. Social Institutions in Charge of Social Common Capital

In each social institution σ, the minimum quantities of factors of production required to provide services of social common capital by a^σ and at the same time to engage in investment activities to accumulate the stock of fixed factors of production by $z^\sigma = (z^\sigma_f)$ with the use of produced goods by $c^\sigma = (c^\sigma_j)$ are specified by a vector-valued function:

$$f^\sigma(a^\sigma, z^\sigma, c^\sigma) = [f^\sigma_f(a^\sigma, z^\sigma, c^\sigma)].$$

For each social institution σ, the production possibility set T^ρ is composed of all combinations $(a^\sigma, z^\sigma, c^\sigma)$ of provision of services of social common capital a^σ, investment z^σ, and use of produced goods c^σ that are possible with the organizational arrangements, technological conditions, and the given endowments of factors of production in social institution σ, K^σ. That is, it may be expressed as

$$T^\sigma = \left\{ (a^\sigma, z^\sigma, c^\sigma) : (a^\sigma, z^\sigma, c^\sigma) \geqq 0, \quad f^\sigma(a^\sigma, z^\sigma, c^\sigma) \leqq K^\sigma \right\}.$$

The following neoclassical conditions are assumed:

(T^{σ} 1) $f^{\sigma}(a^{\sigma}, z^{\sigma}, c^{\sigma})$ are defined, positive, continuous, and continuously twice differ-differentiable with respect to $(a^{\sigma}, z^{\sigma}, c^{\sigma})$ for all $f^{\sigma}(a^{\sigma}, z^{\sigma}, c^{\sigma}) \geqq 0$.

(T^{σ} 2) $f^{\sigma}_{a^{\sigma}}(a^{\sigma}, z^{\sigma}, c^{\sigma}) > 0,\ f^{\sigma}_{z^{\sigma}}(a^{\sigma}, z^{\sigma}, c^{\sigma}) > 0,\ f^{\sigma}_{c^{\sigma}}(a^{\sigma}, z^{\sigma}, c^{\sigma}) < 0$ for all $(a^{\sigma}, z^{\sigma}, c^{\sigma}) \geqq 0$.

(T^{σ} 3) $f^{\sigma}(a^{\sigma}, z^{\sigma}, c^{\sigma})$ are strictly quasi-convex with respect to $(a^{\sigma}, z^{\sigma}, c^{\sigma})$ for all $(a^{\sigma}, z^{\sigma}, c^{\sigma}) \geqq 0$.

(T^{σ} 4) $f^{\sigma}(a^{\sigma}, z^{\sigma}, c^{\sigma})$ are homogeneous of order 1 with respect to $(a^{\sigma}, z^{\sigma}, c^{\sigma})$.

3.5. Capital Accumulation in the Prototype Model of Social Common Capital

The accumulation of the stock of capital goods in private firm ρ is given by the following differential equation

$$\dot{K}_t^{\rho} = z_t^{\rho} - \mu K_t^{\rho},\ K_0^{\rho} = K_o^{\rho}, \tag{14}$$

where z_t^{ρ} is the vector specifying the levels of investment in capital goods in private firm ρ at time t and μ is the rate of depreciation.

Similarly, the accumulation of the stock of capital goods in social institution σ is given by the following differential equation

$$\dot{K}_t^{\sigma} = z_t^{\sigma} - \mu K_t^{\sigma},\ K_0^{\sigma} = K_o^{\sigma}, \tag{15}$$

where z_t^{σ} is the vector specifying the levels of investment in capital goods in social institution σ at time t and μ is the rate of depreciation.

4. Imputed Prices and Sustainable Processes of Capital Accumulation in the Prototype Model of Social Common Capital

Exactly as in the aggregative model of capital accumulation, the imputed price of capital in the prototype model of social common capital is defined. The imputed price, in units of the utility, of each kind of capital at time t, ψ_t, is the discounted present value of the marginal increases in total utility in the future due to the marginal increase in the stock of capital of that kind at time t. When we denote by r_τ the marginal increase in the total utility at future time τ, the imputed price at time t, ψ_t, is given by

$$\psi_t = \int_t^\infty r_\tau e^{-(\delta+\mu)(\tau-t)} d\tau . \qquad (16)$$

By differentiating both sides of (16) with respect to time t, we obtain the following differential equation:

$$\dot{\psi}_t = (\delta + \mu)\psi_t - r_t . \qquad (17)$$

As in the case of the aggregative model of capital accumulation, we suppose that capital is transacted as an asset on a virtual capital market that is perfectly competitive and the imputed price ψ_t is identified with the market price at time t. Consider the situation in which the unit of such an asset is held for the short time period $[t,\ t+\Delta t]\ (\Delta t > 0)$. The gains obtained by holding such an asset are composed of "capital gains" $\Delta\psi_t = \psi_{t+\Delta t} - \psi_t$ and "earnings" $r_t \Delta t$; that is,

$$\Delta\psi_t + r_t \Delta t .$$

On other hand, the costs of holding such an asset for the time period $[t,\ t+\Delta t]$ consist of "interest payments" $\delta\psi_t\Delta t$ and "depreciation charges" $\mu\psi_t\Delta t$, where the social rate of discount δ is identified with the market rate of interest. Hence, on a virtual capital market, these two amounts become equal; that is,

$$\Delta\psi_t + r_t\Delta t = \delta\psi_t\Delta t + \mu\psi_t\Delta t .$$

By dividing both sides of this equation by Δt and taking the limit as $\Delta t \to 0$, we obtain relation (17).

We define that the imputed price ψ_t is at the sustainable level at time t, if it remains stationary at time t; i. e.,

$$\dot{\psi}_t = 0 \quad \text{at time } t,$$

where it may be reminded that $\dot{\psi}_t$ refers to the time derivative with respect to the time of the virtual capital market at time t.

From the basic differential equation (17), the imputed price ψ_t is at the sustainable level at time t, if, and only if,

$$\psi_t = \frac{r_t}{\delta + \mu} \quad \text{at time } t,$$

where r_t is the marginal increase in total utility due to the marginal increase in the stock of capital of that kind at time t.

With respect to the prototype model of social common capital, the imputed price of capital in private firm ρ at time t, ψ_t^ρ, is at the sustainable level at time t, if, and only if,

$$\psi_t^\rho = \frac{r_t^\rho}{\delta + \mu} \quad \text{at time } t, \qquad (18)$$

where r_t^ρ is the marginal increase in total utility due to the marginal increase in the stock of capital in private firm ρ at time t.

Similarly, the imputed price of capital in social institution σ at time t, ψ_t^σ, is at the sustainable level at time t, if, and only if,

$$\psi_t^\sigma = \frac{r_t^\sigma}{\delta + \mu} \quad \text{at time } t, \qquad (19)$$

where r_t^σ is the marginal increase in total utility due to the marginal increase in the stock of capital in social institution σ at time t.

A time-path of capital accumulation is defined *sustainable*, if the imputed prices of all kind of capital, both private capital and social common capital, are at the sustainable levels at all times, i. e., (18) and (19) hold at all times t.

4.1. Sustainable Processes of Consumption and Investment

We presume that the imputed prices of capital goods in private firms and social institutions in charge of social common capital, all at time t, are given, respectively, by ψ_t^ρ and ψ_t^σ. Then the imputed real national income in units of the utility at time t is given by

$$H_t = \sum_\nu u^\nu(c_t^\nu, \varphi^\nu(a_t)a_t^\nu) + \sum_\rho \psi_t^\rho(z_t^\rho - \mu K_t^\rho) + \sum_\sigma \psi_t^\sigma(z_t^\sigma - \mu K_t^\sigma),$$

where c_t^ν is the vector of consumption, and z_t^ρ, z_t^σ are, respectively, the vectors of investment in the capital of private firm ρ and social institution σ, all at time t.

The optimum levels of consumption and investment at time t, c_t^ν, z_t^ρ, z_t^σ, are obtained as the solution for the following maximum problem.

Maximum Problem. Maximize the imputed real national income in units of the utility at time t, H_t, subject to the feasibility constraints:

$$\sum_\nu c_t^\nu + \sum_\sigma c_t^\sigma \leqq \sum_\rho x_t^\rho \tag{20}$$

$$\sum_\nu a_t^\nu + \sum_\rho a_t^\rho \leqq a_t \tag{21}$$

$$a_t \leqq \sum_\sigma a_t^\sigma \tag{22}$$

$$f^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho) \leqq K_t^\rho \tag{23}$$

$$f^\sigma(a_t^\sigma, z_t^\sigma, c_t^\sigma) \leqq K_t^\sigma, \tag{24}$$

where a_t^ν, a_t^ρ are, respectively, the amounts of services of social common capital used by individuals ν and private firms ρ, a_t^σ is the amount of services of social common capital provided by social institutions σ, and a_t is the total amount of services of social common capital, all at time t.

Let L_t be the Lagrangian form for this maximum problem:

$$L_t = \sum_\nu u^\nu(c_t^\nu, \varphi^\nu(a_t)a_t^\nu) + \sum_\rho \psi_t^\rho(z_t^\rho - \mu K_t^\rho) + \sum_\sigma \psi_t^\sigma(z_t^\sigma - \mu K_t^\sigma)$$

$$+ p_t\left[\sum_\rho x_t^\rho - \sum_\nu c_t^\nu - \sum_\sigma c_t^\sigma\right] + \theta_t\left[a_t - \sum_\nu a_t^\nu - \sum_\rho a_t^\rho\right] + \pi_t\left[\sum_\sigma a_t^\sigma - a_t\right]$$

$$+ \sum_\rho r_t^\rho\left[K_t^\rho - f^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho)\right] + \sum_\sigma r_t^\sigma\left[K_t^\sigma - f^\sigma(a_t^\sigma, z_t^\sigma, c_t^\sigma)\right],$$

where $p_t, \theta_t, \pi_t, r_t^\rho, r_t^\sigma$ are, respectively, the Lagrangian unknowns associated with constraints (20), (21), (22), (23), and (24).

The optimum conditions are characterized by the following marginality conditions, in addition to the feasibility conditions (20)–(24):

$$u_{c^\nu}^\nu(c_t^\nu, \varphi^\nu(a_t)a_t^\nu) \leqq p_t \qquad (\text{mod. } c_t^\nu) \tag{25}$$

$$u_{a^\nu}^\nu(c_t^\nu, \varphi^\nu(a_t)a_t^\nu)\varphi^\nu(a_t) \leqq \theta_t \qquad (\text{mod. } a_t^\nu) \tag{26}$$

$$p_t \leqq r_t^\rho f_{x^\rho}^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho) \qquad (\text{mod. } x_t^\rho) \tag{27}$$

$$\psi_t^\sigma \leqq r_t^\rho f_{z^\rho}^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho) \qquad (\text{mod. } z_t^\rho) \tag{28}$$

$$\theta_t \geqq r_t^\rho\left[- f_{a^\rho}^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho)\varphi^\rho(a_t)\right] \qquad (\text{mod. } a_t^\rho) \tag{29}$$

$$f^\rho(x_t^\rho, z_t^\rho, \varphi^\rho(a_t)a_t^\rho) \leqq K_t^\rho \qquad (\text{mod. } r_t^\rho) \tag{30}$$

$$\pi_t \leqq r_t^\sigma f_{a^\sigma}^\sigma(a_t^\sigma, z_t^\sigma, x_t^\sigma) \qquad (\text{mod. } a_i^\sigma) \tag{31}$$

$$\psi_t^\sigma \leqq r_t^\sigma f_{z^\sigma}^\sigma(a_t^\sigma, z_t^\sigma, x_t^\sigma) \qquad (\text{mod. } z_i^\sigma) \tag{32}$$

$$p_t \geqq r_t^\sigma\left[- f_{x^\sigma}^\sigma(a_t^\sigma, z_t^\sigma, x_t^\sigma)\right] \qquad (\text{mod. } x_t^\sigma) \tag{33}$$

$$f^\sigma(a_t^\sigma, z_t^\sigma, x_t^\sigma) \leqq K_t^\sigma \qquad (\text{mod. } r_t^\sigma) \tag{34}$$

$$\theta_t - \pi_t = \tau_t \pi_t, \quad \tau_t = \frac{\tau(a_t)a_t}{1-\tau(a_t)a_t}. \tag{35}$$

Lagrange unknowns p_t , θ_t , π_t may be interpreted, respectively, as the imputed prices of the output, the prices for the use of social common capital, and the prices paid for the provision of services of social common capital, and r_t^ρ and r_i^σ are, respectively, the imputed rents of capital in private firm ρ and social institution σ, whereas ψ_t^ρ and ψ_t^σ are, respectively, the imputed prices of real the capital in private firm ρ and social institution σ, all at time t, measured in units of the utility.

A simple calculation shows that the imputed rents of capital in private firm ρ and social institution σ, r_t^ρ and r_i^σ, are, respectively, the marginal increases in total utility due to the marginal increases in the stock of capital in private firm ρ and social institution σ, both at time t. Hence, the sustainable processes of consumption and investment in the prototype model of social common capital may be obtained when the imputed prices of capital in private firm ρ and social institution σ at time t, ψ_t^ρ

and ψ_t^σ are, respectively, equal to the discounted present values of the imputed rents of capital in private firm ρ and social institution σ assuming that the imputed rents of capital remain stationary; i., e., the following relations hold for all private firms ρ and social institutions σ at all times t:

$$\psi_t^\rho = \frac{r_t^\rho}{\delta + \mu}, \qquad \psi_t^\sigma = \frac{r_t^\sigma}{\delta + \mu}. \tag{36}$$

The relations (36) mean that the stationary expectations hypothesis holds true as regards the future schedule concerning marginal efficiency of investment of all kind of capital, private capital and social common capital.

4.2. Sustainable Processes of Capital Accumulation and Market Equilibrium

The optimum conditions for the sustainable processes at time t, as obtained above, are identical with those for market equilibrium at time t, when the imputed prices of capital in private firm ρ and social institution σ, ψ_t^ρ and ψ_t^σ are, respectively, regarded as the "market prices" of capital in private firm ρ and social institution σ, respectively, all at times t, assuming that the stationary expectations hypothesis holds true as regards the future marginal efficiency of investment of all kind of capital, and the social common capital taxes are levied upon the use of services of social common capital.

Indeed, the optimum conditions (25) – (35), together with the feasibility conditions (20) – (24), precisely correspond to the conditions for the market equilibrium in the model of social common capital at time t:

(i) Each individual ν chooses the combination (c_t^ν, a_t^ν) of consumption c_t^ν and the use of services of social common capital a_t^ν so that the individual ν's utility

$$u^\nu(c_t^\nu, \varphi^\nu(a_t)a_t^\nu)$$

is maximized subject to the budget constraint

$$p_t c_t^\nu + \theta_t a_t^\nu = y_t^\nu,$$

where y_t^ν is the income of individual ν.

(ii) Each private firm ρ chooses the combination $(x_t^\rho, z_t^\rho, a_t^\rho)$ of production x_t^ρ, investment z_t^ρ, and the use of services of social common capital a_t^ρ in such a manner that net profits

$$p_t x_t^\rho + \psi_t^\rho z_t^\rho - \theta_t a_t^\rho$$

are maximized over $(x_t^\rho, z_t^\rho, a_t^\rho) \in T^\rho$.

(iii) Each social institution σ chooses the combination $(a_t^\sigma, z_t^\sigma, x_t^\sigma)$ of the provision of services of social common capital a_t^ρ, investment z_t^σ, and the use of produced goods x_t^σ in such a manner that net profits

$$\pi_t a_t^\sigma + \psi_t^\sigma z_t^\sigma - p_t x_t^\sigma$$

are maximized over $(a_t^\sigma, z_t^\sigma, x_t^\sigma) \in T^\rho$.

(iv) At the prices p, total demand for goods are equal to total supply:

$$\sum_\nu c_t^\nu + \sum_\sigma c_t^\sigma = \sum_\rho x_t^\rho .$$

(v) At the prices for the provision and the use of services of social common capital, π_t and θ_t, the total amounts of the provision and use of services of social common capital are equal:

$$\sum_\nu a_t^\nu + \sum_\rho a_t^\rho = \sum_\sigma a_t^\sigma .$$

(vi) Social common capital taxes at the rate τ_t are levied upon the use of services of social common capital; i. e.

$$\theta_t - \pi_t = \tau_t \pi_t, \quad \tau_t = \frac{\tau(a_t)a_t}{1-\tau(a_t)a_t}.$$

(vii) The expectations concerning future marginal productivity of capital of all kinds are stationary, i.e.,

$$\psi_t^\rho = \frac{r_t^\rho}{\delta+\mu}, \quad \psi_t^\sigma = \frac{r_t^\sigma}{\delta+\mu},$$

where ψ_t^ρ, ψ_t^σ and r_t^ρ, r_t^σ are respectively the imputed prices and the rental prices of the capital goods accumulated in private firm ρ and social institution σ.

The discussion above may be summarized as

Proposition 2. *In the prototype model of social common capital, the optimum conditions for the sustainable time-path of consumption and accumulation of private capital and social common capital coincide precisely with those for market equilibrium with the following assumptions:*

(i) *The social common capital taxes at the rate* τ *are levied so that*

$$\theta - \pi = \tau\pi, \quad \tau = \frac{\tau(a)a}{1-\tau(a)a}$$

where π, θ *are, respectively, the price paid for the provision of services of social common capital and the price charged to services of common capital, and* $\tau(a)$ *is the impact coefficient with respect to the use of services of social common capital.*

(ii) *The expectations concerning future marginal productivity of capital of all kinds are stationary.*

Thus the sustainable processes of consumption and capital accumulation, including both private capital and social common capital, are obtained solely in terms of the state of the economy at each moment in time, independent of the hypotheses concerning the future schedules of marginal efficiency of investment in private capital and social common capital.

On the other hands, the dynamically optimum processes of consumption and capital accumulation, including both private capital and social common capital, are obtained only at the hypothesis of perfect foresight concerning the future schedules of marginal efficiency of investment in all kinds of capital, as in detail discussed, e. g., in Uzawa (2003, 2005).

Remark: It may be noted that the analysis of sustainable processes of capital accumulation for the prototype model of social common capital developed in the present paper holds true for the general circumstances in which the relevant functions, such as utility functions, production functions, and Penrose functions, may change over time.

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