DBJ Research Center on Global Warming
Discussion Paper Series No. 53 (10/2015)

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This paper is a preliminary material in the draft form to stimulate discussion and comments from academics and policy makers. Views expressed in this paper are those of the author(s) and do not necessarily reflect the official views of the Development Bank of Japan.

The Revelation of the Time Preference Rate and Intertemporal Negative Externality

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Abstract

Different from the overlapping-generations model, it is allowable to discount the future utility in a dynasty model without intergenerational conflicts. While the social utility discount rate should be equal to the time preference rate, such a rate is unobservable and hence must be estimated based on observables and measurables based on a theory. Much precedent research uses the Ramsey type optimal growth theory for this estimation. However one must note almost all estimations neglect the existence of intertemporal negative externalities. This problem is vital when one analyzes the global warming problem mainly caused by the excess concentration of carbon dioxide (CO_2). This is because there emerges another effect of capital accumulation besides the improvement of product capacity that is expressed by the rate of interest (or equivalently, marginal productivity of capital). There exists the negative externality to the future productivity that is originated from the excess emissions of CO_2. Accordingly, the rate of time preference is always lower than the rate of interest even in a sustainable growth path where there is no growth in consumption.

Keywords: Time Preference; Interest Rate; Negative Externality by the Emissions of CO_2

JEL Classifications: Q54;Q58;D61;D62

1. Introduction

It requires some elaborations to determine the rate of time preference in a dynastic social planning problem because such a rate is unobservable. Determining its value equivalently means to find a substitutable form for the rate of time preference by observable and/or measurable variables such as the rate of interest. Many preceding studies[1] are based on the Ramsey type optimal growth theory for the estimation. Whenever the rate of preference exceeds the rate of interest, the optimal consumption proportionately decreases because current consumption is more advantageous than the future. Accordingly the following well-known formula is obtained:

$$\rho = r - \eta \frac{\dot{c}}{c} \qquad (1)$$

where ρ is the rate of time preference, and r is the (real) rate of interest. η and $\frac{\dot{c}}{c}$ denote

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[1] In discounting the far-distant future events, two approaches are noted (Arrow, et al. 1996). One is the descriptive approach, which typically focus on the opportunity cost of capital whose level is observable as the rate of return on alternative investments in the market (Nordhaus, 2013). The advocates of this approach employ the opportunity cost of capital as the discount rate since they claim that investments in reducing climate change must compete with alternative investments in the market. See also Nordhaus (2007) and Weitzman (2007). The other is the prescriptive approach, where the advocates maintain the view that the market interest rates fail to indicate the trade-offs of consumptions across generations and that the discount rate should be derived from ethical point of view. Those advocates include Cline (1992) and Stern (2007).

Proponents in both approaches use the same Ramsey rule of Equation (1) in interpreting their discount rates. As we argue later in the article, main problem of this expression is the fact that there is no negative externality postulated in the rule. (See Appendix I to find the limitation of the standard Ramsey rule. See also Appendix II for its scope of validity in relation with the technological change.)

In addition, this formulation does not provide the solutions for the time preference rates ρ or elasticity of marginal consumption η endogenously, leaving the each advocate in both approaches to assign appropriate values as parameters according to his/her observations or value judgement.

Examples of other approaches where social discount rate is endogenously derived are found in Otaki (2013 and 2015).

the elasticity of marginal (instantaneous) utility[2] and the increase rate of consumption, respectively. According to Equation (1), except for a stationary state, the time preference rate is lower than the rate of interest as far as an economy grows. Those who infer that the time preference in a social planning might be lower than the market interest rate focus on refining the estimation of the second term of Equation (1)[3].

However, when one analyzes the efficient allocation of public bads such as the emissions of CO_2, he or she finds that Equation (1) alienates from the effect caused by such intertemporal negative externalities. This asks us for an extension of the theory.

Suppose that a unit current capital investment not merely strengthens the production capacity of an individual firm but also, via the aggravation of the global warming, if not offsetting the former positive effect, lowers the productivity as a whole. Let this rate be denoted ψ. In this case, the social rate of return is less than the rate of interest that is equal to the marginal productivity of capital of an individual firm.

This study obtains the following modified formula:

$$\rho = r - \psi^* \left[\nu \frac{\dot{K}_t}{K_t} \right] - \eta \frac{\dot{c}_t}{c_t} \tag{2}$$

where ψ^* is the optimal carbon tax rate that is solved in the next section. ν denotes the elasticity of CO_2 stock to capital accumulation. g_n is the autonomous emissions associated with the incumbent capital equipment. Thus, one finds that the time preference ρ is lower than the rate of interest r even in a stationary state where $\frac{\dot{c}}{c} = 0$.

This article is organized as follows. Section 2 derives the formula in Equation (2) and

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[2] Three kinds of interpretation of the elasticity of marginal utility are known (Stern, 2008). They are intratemporal distribution, intertemporal distribution and attitudes to risk, respectively. Directly related to our argument is the intertemporal distribution and in this analysis we postulate that we can set the appropriate values for η. Since, under the stationary state where $\frac{\dot{c}}{c} = 0$, the role of η plays the secondary role, we leave how to treat η to a future study.

[3] Introduction of uncertainty into discounting leads us to a theory of declining discount rates (Arrow, et al., 2013). For example, uncertainty of the growth rates of future consumption (See Gollier, 2012) and uncertainty of future discount rates such as the rates of return on investment (See Weitzman, 1998 and Gollier-Weitzman, 2010) will typically cause the discount rates to decline as time goes by.

discusses the implications. Section 3 concludes with brief remarks.

2. The Model

2.1 The Derivation of the New Formula

Since Equations (1) and (2) are derived from the local maximization on the optimal path, it is sufficient to consider the optimality condition between two sequential periods. Let the aggregate production function over the world, f ,be denoted[4]

$$y_t \equiv f\left(K_t, E_t\right), E_t \equiv \varphi\left(K_t\right), \quad \frac{\partial f}{\partial K} > 0, \frac{\partial f}{\partial E} < 0, \frac{\partial \varphi}{\partial K} > 0. \tag{3}$$

y_t denotes the volume of current output, K_t, E_t are the capital stock and the accumulated emissions of CO_2 that prescribes the environmental condition, respectively. φ is the function that represents how much emissions are accumulated by production activities by using capital.[5]

Consequently, the social planning problem which one must solve is

$$\max_{I_t} \left[u\left(f\left(K_t^*, \varphi\left(K_t^*\right)\right) - I_t \Delta t \right) + \frac{1}{1+\rho\Delta t} u\left(f\left(K_{t+\Delta t}, \varphi\left(K_{t+\Delta t}\right)\right) - I_{t+\Delta t}^* \Delta t \right) \right] \tag{4}$$

subject to

$$K_{t+\Delta t} = K_t^* + I_t \cdot \Delta t, \tag{5}$$

where u is a strictly concave utility function. K_t^* is the optimal capital stock at the beginning of period t that has been determined beforehand by the past decisions. $I_{t+\Delta t}^*$ is the given optimal investment during period $t+\Delta t$. ρ denotes the instantaneous time preference rate.

Since the productivity slowdown brought about the CO_2 emissions is external as for each firm, the profit-maximization condition requires

$$\frac{\partial f}{\partial K} = \left[1 + r \cdot \Delta t\right] \tag{6}$$

where r is the instantaneous interest rate. In addition, if the proportional carbon tax to the increase rate of CO_2 stock, the rate of which is ψ , is levied to firms to internalize the

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[4] We assume that each economy in the world has the same production and utility function, and endowments. Accordingly, such functions can be aggregated.

[5] We do not exclude the possibility that an economy-wide carbon neutral technology might be available, which effectively absorbs emitted CO_2, and/or the usage of non-carbon emitting energy resources fully in order to avoid the increase of the CO_2 concentration.

negative externalities originated from CO_2 emissions, this lowers the return from the new capital investment by $\psi \cdot \frac{\Delta E}{E\Delta t}\Delta t$. The optimal tax rate ψ should be equalized to the marginal negative productivity of capital brought by the emissions. Thus, one obtains

$$\begin{aligned}&\frac{\partial f}{\partial \varphi}\cdot\left[\frac{\partial \varphi}{\partial K}\frac{\Delta K}{\Delta t}\right]=-\psi^{*}\frac{\Delta E}{E\Delta t}\\&\Leftrightarrow \frac{\partial f}{\partial \varphi}\cdot\left[\frac{K\partial \varphi}{\varphi\partial K}\frac{\Delta K}{K\Delta t}\right]\Delta t=-\psi^{*}\frac{\Delta E}{\Delta t}\Delta t=-\psi^{*}\left[\frac{K\partial \varphi}{\varphi\partial K}\frac{\Delta K}{K\Delta t}\right]\Delta t\end{aligned} \tag{7}$$

Thus, the optimal tax rate ψ^{*} is equal to how many outputs are curtailed by one percent increase in the accumulate stock of CO_2. That is, it is evident from Equation (7) that

$$\psi^{*}=-\frac{\partial f}{\partial \varphi} \tag{8}$$

holds.

Furthermore, one must note that the following relationships:

$$o(\Delta K)=o\left(\frac{\Delta K}{\Delta t}\cdot\Delta t\right)=o(\Delta t),\ \ o(\Delta c)=o\left(\frac{\Delta c}{\Delta t}\Delta t\right)=o(\Delta t). \tag{9}$$

Differentiating Equation (4) with respect to I_t, and taking the relationships in Equations (6), (7) and (8) into consideration, we obtain

$$\begin{aligned}&1+\rho\Delta t=\left[1+r\Delta t-\psi^{*}\frac{\Delta E}{\Delta t}\Delta t\right]\frac{u'\left(c_{t+\Delta t}^{*}\right)}{u'\left(c_{t}^{*}\right)}\\&=\left[1+\left[r-\psi^{*}\left[\frac{K\partial \varphi}{\varphi\partial K}\frac{\Delta K}{K\Delta t}\right]+\frac{o(\Delta K)}{\Delta t}+\frac{o(\Delta t)}{\Delta t}\right]\Delta t\right]\left[1-\eta\frac{\Delta c}{c\Delta t}\Delta t+o(\Delta c)\right]\end{aligned} \tag{10}$$

Dividing both sides of Equation (9) by Δt, one can ascertain

$$\rho=r-\psi\left[\frac{K\partial \varphi}{\varphi\partial K}\frac{\Delta K}{K\Delta t}\right]-\eta\frac{\Delta c}{c\Delta t}+\frac{o(\Delta t)}{\Delta t}. \tag{11}$$

Taking the limit $\Delta t\to 0$ of Equation (11), finally we obtain the following representation form:

$$\rho=r-\psi^{*}\left[\nu\frac{\dot{K}_t}{K_t}\right]-\eta\frac{\dot{c}_t}{c_t} \tag{12}$$

where ν is the elasticity of the accumulated CO_2 to capital stock[6].

[6] For example, if the form of the production function is

2.2 The Implications of the Derived Formula

First, we shall consider the economic meaning of the formula in Equation (12). The most prominent feature of the formula is that the social rate of return from unit capital accumulation is lower than the real interest rate r. This is because an additional capital reduces productivity via the accumulation of CO_2 emissions. This negative effect appears in the second term of the right-hand side of Equation (12). The inside of the bracket comprises the total increase rate of emitted CO_2. Since ψ^* is determined optimally in order that it completely internalizes such a negative diseconomy, the second term, as a whole, corresponds to the exact carbon price that has to been paid. Thus, the social rate of return is lowered by $\psi^*\left[\nu \frac{\dot{K}_t}{K_t}\right]$ in comparison with the market interest rate. Accordingly, the time preference rate ρ is strictly smaller than the rate of interest, r, even in the stationary state where $\frac{\dot{c}_t}{c_t}=0$ holds. Such a constant consumption path is dubbed *sustainable* by Dasgupta and-Heal (1974) and Solow (1986) [7]. This definition of sustainability seems to be characterized by the intergenerational fair opportunity for consumption, although the analysis is formulated as the optimization problem of representative individual and firm, both of which live infinitely.[8]

From Equation (12), the capital growth rate, g_k^*, which sustains the zero consumption growth is expressed as

$$g_k^* = \frac{r-\rho}{\psi^* \nu}. \qquad (13)$$

Equation (13) indicates that the sustainable growth path is affected by the following two factors, given that the difference $r-\rho$ is positive.

First, when the optimal carbon tax rate ψ^* is high, which implies that the

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$$y_t(\omega) = f\left(k_t(\omega), \varphi(K_t,t)\right) = \left[\varphi(K_t,t)\right]\left[k_t(\omega)\right]^{\alpha}, 0<\gamma<\alpha\le 1,$$
$$\varphi(K_t,t) = \left[K_t\right]^{-\nu}, K_t \equiv \int_0^1\left[k_t(\omega)\right]d\omega,$$

and the utility function belongs to the CRRA family, the formula in Equation (12) becomes an exact solution.

[7] Note that Solow (1992) is skeptical about the arbitrariness of the concept of sustainability.

[8] Find Appendix III to see if there exists a feasible path with a constant consumption growth.

accumulation of CO_2 exacerbates the productivity more seriously, the sustainable growth rate becomes lower. This is because a higher tax rate curtails the social rate of return from capital. Second, if the elasticity of CO_2 emissions to capital investment ν is high, the growth rate also becomes lower. This result comes from the fact that a higher elasticity substantially raises the carbon price.

2.3 The Limit of Economic Growth: The Case of Linear Homogenous Individual Production Function with Negative Externality

We heretofore assume that the rate of interest, r, is constant over time. However, as Equation (6) shows, this varies along with capital accumulation. Since we assume that

$$\frac{\partial^2 f}{\partial K^2} + \frac{\partial^2 f}{\partial K \partial E}\frac{dE}{dK} < 0, \qquad (14)$$

the relationship between capital stock, K, and the rate of interest, r, becomes downward sloping curve KK as illustrated in Figure 1. It is clear from Equation (13) that if the world interest rate is located above Line LL, capital accumulation advances. Otherwise, disinvestment should be fastened. Ultimately, the world economy converges to Point E. This asserts that there is a limit of the world economy's growth when a intertemporal negative externality such as the global warming exists.

What one must note is that even though the production function of each economy is a linear function on capital stock, which means

$$\frac{\partial^2 f}{\partial K^2} = 0, \qquad (15)$$

Equation (14) holds, and thus the property of Figure 1 is preserved. This means that our atmosphere is not renewable resources as human being as a whole, and therefore, even though the artificial technological system in the short run contains no non-renewable resources, the scarcity of the quality of atmosphere becomes the bottleneck of the production/consumption enjoyment in the long run.

3. Concluding Remarks

The representation theory concerning the time preference rate under an intertemporal negative externality (e.g., caused by the excess emissions of CO_2) has been considered with the following results. First, even under a sustainable equilibrium where the aggregate consumption level is kept constant, the time preference rate is lower than the interest rate. This stems from the fact that the negative externality associated with the capital accumulation lowers the social rate of return from capital. It is notable that such a negative effect is not taken into consideration in the preceding articles analyzing the global warming problem despite of its importance.

Second, the optimal rate of the proportional carbon tax is also derived explicitly. The rate is equal to the marginal decrease in product per one percentage increase in CO_2 in atmosphere. That is, if the emissions exacerbate productivity more seriously, the tax rate should be heightened.

Finally, besides the optimal carbon tax rate, the following economic factor affects the social rate of return from capital that is defined at the sustainable stationary state in the sense of Dasgputa-Heal and Solow: the elasticity of the density of CO_2 to the capital accumulation ν. Whenever ν increases, which means that more CO_2 is emitted by a unit capital accumulation (i.e., lower efficiency in emissions reduction process), lowers the social rate of return from capital because this incurs higher costs for capital accumulation.

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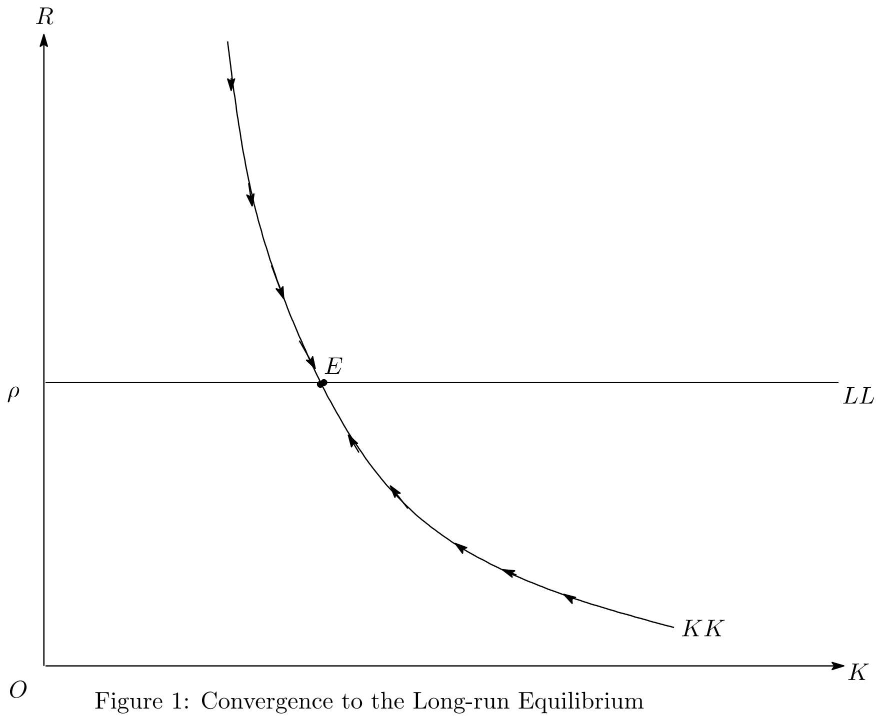

Figure 1: Convergence to the Long-run Equilibrium

Appendix I: The Formula $\rho = r - \eta \frac{\dot{c}}{c}$ Cannot Describe the Global Warming Problem Intact

The Ramsey formula $\rho = r - \eta \frac{\dot{c}}{c}$ is a special case of our formula (2) which corresponds to the case that $\psi^* \nu = 0$ and $\frac{\partial F}{\partial K} = r = const.$ One must note that since the production function of each industry is linear on capital, accordingly, the aggregation is feasible. Thus, we obtain the macroeconomic production function F as

$$y_t = rK_t \qquad \text{(A.1)}$$

Equation (A.1) implies that there is no other scarce production resource than capital. This fact asserts that there is a serious drawback in the standard Ramsey model to analyze the global warming problem.

This is because the scarcity, which stems from the quality of our atmosphere (measured by the mass of CO_2 contained in the atmosphere), is the vital issue in this problem. As such, unless we assume some optimistic exogenous emissions-absorbing technological progress, it is a plausible theoretical formulation that the aggregated production function is subject to the diminishing return to scale owing to the Marshallian negative externality from CO_2 emissions. That is, theoretically, the congestion in availing the atmosphere is the very core of the global warming problem.

Appendix II: The Formula $\rho = r - \eta \frac{\dot{c}}{c}$ and Technological Progress

Preserving the relevance of the formula $\rho = r - \eta \frac{\dot{c}}{c}$ in the context of the global warming problem, we need to introduce some exogenous technological progress that reduces the concentration of CO_2 at a constant rate β. In such a case, for example, a macroeconomic production becomes

$$Y_t = K_t \left[\left(K_t e^{-\beta t} \right)^{-\nu} \right]. \qquad \text{(A.2)}$$

The inside of the bracket indicates the external effect associated with the concentration/reduction of CO_2, which corresponds to the exogenous total factor productivity (TFP) for each firm. Accordingly, as far as the TFP is kept constant, the world economy can achieve a steady growth rate because the macroeconomic production function becomes linear.

By (A.2) such a rate is represented as

$$\frac{dK_t e^{-\beta t}}{dt} = 0 \Rightarrow \frac{\dot{K}_t}{K_t} = \beta , \qquad \text{(A.3)}$$

$$F'\left(K^*\right) = r^* = const.. \qquad \text{(A.4)}$$

Applying Equations (A.3) and (A.4) to the formula (2), one obtains

$$\rho = r^* - \eta \frac{\dot{c}}{c} . \qquad \text{(A.5)}$$

One here must note that Equation (A.5) presumes an interior solution in the sense that the increase rate of consumption, $\frac{\dot{c}_t}{c_t}$, can take an arbitrary value. However, from the budget constraint,

$$\begin{aligned} &\dot{K}_t = \beta K_t = r^* K_t - c_t \\ &\Rightarrow K_t = \frac{c_t}{r^* - \beta} \Rightarrow \beta = \frac{\dot{K}_t}{K_t} = \frac{\dot{c}_t}{c_t} \end{aligned} \qquad \text{(A.6)}$$

holds. Thus, (A5) necessarily becomes at the steady state equilibrium such as

$$\rho = r - \eta\beta . \qquad \text{(A.7)}$$

In other words, Equation (A.7) implies that the feasibility of the future increase in consumption is entirely relies on the autonomous technological progress in the reduction of the stock of CO_2.

The stability of the stationary state can be proved as follows. Again, from the budget constraint,

$$\begin{aligned} &\dot{K}_t = g_t K_t = r_t K_t - c_t \\ &\Rightarrow K_t = \frac{c_t}{r_t - g_t} \Rightarrow g_t + \frac{\dot{g}_t}{\left[r_t - g_t\right]} = \frac{\dot{c}_t}{c_t} \end{aligned} \qquad \text{(A.8)}$$

holds.

On the other hand, since

$$r_t = \left[K_t e^{-\beta t}\right]^{\nu} , \qquad \text{(A.9)}$$

logarithmically differentiating Equation (A.9) with respect to time, one obtains

$$\frac{\dot{r}_t}{r_t} = \nu\left[\beta - g_t\right] . \qquad \text{(A.10)}$$

Thus, our economy can be described by the two differential equations on $\left(g_t, r_t\right)$ as in Equations (A.8) and (A.10). The phase diagram of this system is illustrated in Figure

A.1. Accordingly, the economy has saddle-point property, and the planning economy can achieve these stable paths, which converges to the stationary equilibrium, E. Thus, equilibrium growth rate of consumption in the steady state is equal to β, and the formula (A.7) is upheld.

Appendix III: There Is No Feasible Path with a Positive Constant Consumption Growth Rate

In this Appendix III, we show that any positive constant consumption growth rate path is unfeasible in our model, and hence the only feasible path is the sustainable path (i.e. $\dot{c}=0$).

Assume that the growth rate is denoted g_c. Then Equation (12) implies that there must exist K^* in the stationary state such as

$$\eta g_c = r\left(K^*\right) - \rho > 0 \qquad \text{(A.11)}$$

However,

$$\dot{c} = F'\left(K\right)\dot{K} - \ddot{K} = 0 \qquad \text{(A.12)}$$

holds, if there is a constant K^* which satisfies Equation (A.11). This is a contradiction.

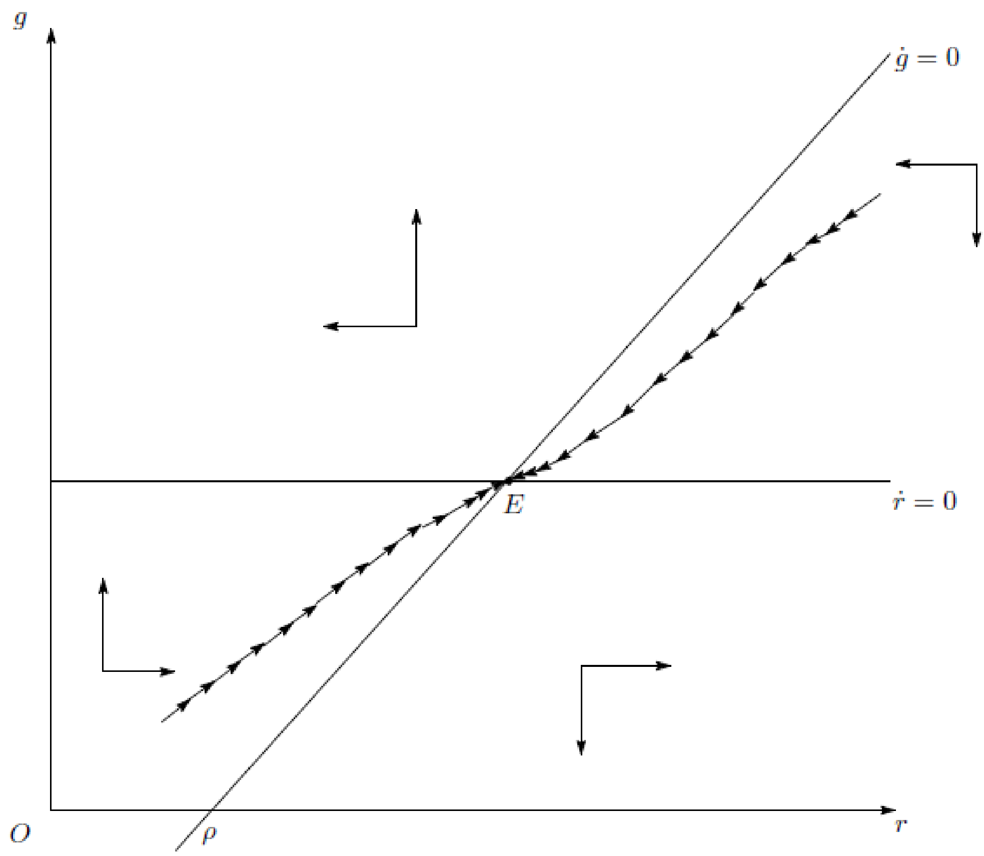

Figure A-1: The Phase Diagram